Commutation relations of vertex operators related with the spin representation of $U_q(D_n^{(1)})$

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1. INTRODUCTION

A new scheme to study the XXZ spin chain in massive regime is introduced by Davies et al in [1]. In this scheme, which is called "quantum symmetry approach", the space of states is identified with some $U_q(A_1^{(1)})$-module. Under this identification, the XXZ Hamiltonian can be constructed on this module as an operator which commutes with the action of $U_q(A_1^{(1)})$. Furthermore, the row transfer matrix and creation (annihilation) operators can be constructed by using vertex operators.

In a similar way to [1], the higher spin chain [2] and the models related to the vector representation of $U_q(A_n^{(1)})$ [3], $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ [5] are studied. In this paper, we consider the model related with the spin representation of $U_q(D_n^{(1)})$.

In physics the model that we consider here is explained as follows. It is a one dimensional quantum spin chain model constructed from the spin representation of $U_q(D_n^{(1)})$. The Hamiltonian acts on the space of the infinite tensor product

$$\cdots \otimes V_{k+1}^{(n)} \otimes V_k^{(n)} \otimes V_{k-1}^{(n)} \otimes \cdots , \quad (k \in \mathbb{Z}),$$

where $V_k^{(n)}$ denotes a copy of spin representation $V^{(+)}$(see Section 2.4). The explicit form of the Hamiltonian is given by

$$\mathcal{H} = \sum_{k\in Z} h_{k+1,k}, \quad h_{k+1,k} = \cdots \otimes \mathrm{id}_{V_{k+2}^{(n)}} \otimes h \otimes \mathrm{id}_{V_{k-1}^{(n)}} \otimes \cdots ,$$

$$h = -(q-q^{-1}) z \frac{d}{dz} PR(z)|_{z=1},$$

[1]This work was partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists and Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture.

where $h_{k+1,k}$ acts non-trivially only on the $(k+1)$-th and the k-th component. The operator $R(z)$ is the R-matrix

$$R(z_1/z_2) \ : \ V_{z_1}^{(n)} \otimes V_{z_2}^{(n)} \longrightarrow V_{z_1}^{(n)} \otimes V_{z_2}^{(n)},$$

and P denotes the transposition i.e. $P(v \otimes w) = w \otimes v$.

In quantum symmetry approach, the problem is formulated as follows. Consider the space of states as $U_q(D_n^{(1)})$-module

$$\bigoplus_{\lambda,\mu} V(\lambda) \otimes V(\mu)^{*a},$$

where λ and μ are level one dominant integral weights, $V(\lambda)$ and $V(\lambda)^{*a}$ denote the irreducible highest weight module with highest weight λ and its antipode dual (cf. [1]). We use vertex operators associated to $U_q(D_n^{(1)})$-modules $V_z^{(k)}$ in order to construct the transfer matrix (the Hamiltonian is obtained from the transfer matrix $T(z)$ as follows: $\mathcal{H} = -(q-q^{-1}) z d \log T(z)/dz|_{z=1}$) and creation (annihilation) operators. Here $V_z^{(k)}$ is the affinization of $V^{(k)}$ that is the following representation of the "derived subalgebra" $U_q'(D_n^{(1)})$ (see section 2):

$$\begin{cases} V^{(1)} & \text{the vector representation,} \\ V^{(k)} \ (k = 2, \cdots, n-2) & \text{the fusion representation,} \\ V^{(n-1)}, \ V^{(n)} & \text{the spin representation,} \end{cases} \qquad (1)$$

The row transfer matrix and creation (annihilation) operators are constructed by using vertex operators for $V_z^{(k)}$ and its dual modules $V_z^{(k)*a^{\pm 1}}$. Vertex operators for these modules are defined by the following $U_q(D_n^{(1)})$-homomorphisms:

$$\begin{aligned} &(\text{type I}) \qquad \Phi_\lambda^{\mu\ V^{(k)}}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z^{(k)}, \\ &(\text{type II}) \qquad \Phi_\lambda^{V^{(k)}\ \mu}(z) : V(\lambda) \longrightarrow V_z^{(k)} \otimes V(\mu). \end{aligned}$$

$$\begin{aligned} &(\text{type I}) \qquad \Phi_\lambda^{\mu\ V^{(k)*a^{\pm 1}}}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z^{(k)*a^{\pm 1}}, \\ &(\text{type II}) \qquad \Phi_\lambda^{V^{(k)*a^{\pm 1}}\ \mu}(z) : V(\lambda) \longrightarrow V_z^{(k)*a^{\pm 1}} \otimes V(\mu). \end{aligned}$$

Commutation relations of vertex operators give us commutation relations of the transfer matrix and creation (annihilation) operators, and then the excitation spectra of the Hamiltonian $\mathcal{H}$. In fact, we can show that vertex operators have the following commutation relations:

$$\begin{aligned} \Phi_{\mu'}^{\nu\ V^{(n)}}(z_2) \Phi_\lambda^{V^{(k)}\ \mu'}(z_1) &= \tau^{(k)}(z_1/z_2) \Phi_\mu^{V^{(k)}\ \nu}(z_1) \Phi_\lambda^{\mu\ V^{(n)}}(z_2), \\ \Phi_{\mu'}^{\nu\ V^{(n)}}(z_2) \Phi_\lambda^{V^{(k)*a^{-1}}\ \mu'}(z_1) &= \tau^{(k)}(z_1/z_2)^{-1} \Phi_\mu^{V^{(k)*a^{-1}}\ \nu}(z_1) \Phi_\lambda^{\mu\ V^{(n)}}(z_2). \end{aligned} \qquad (2)$$

$$\tau^{(k)}(z)=\begin{cases} z^{-\frac{k}{2}}\displaystyle\prod_{j=1}^{k}\frac{\Theta_{q^{4n-4}}(-(-q)^{k+n-2j}z)}{\Theta_{q^{4n-4}}(-(-q)^{k+n-2j}z^{-1})} & (1\le k\le n-2),\\ z^{-\frac{n-2}{4}}\displaystyle\prod_{j=1}^{[\frac{n-1}{2}]}\frac{\Theta_{q^{4n-4}}(-(-q)^{4i-1}z)}{\Theta_{q^{4n-4}}(-(-q)^{4i-1}z^{-1})} & (k=n-1),\\ z^{-\frac{n}{4}}\displaystyle\prod_{j=1}^{[\frac{n}{2}]}\frac{\Theta_{q^{4n-4}}(-(-q)^{4i-3}z)}{\Theta_{q^{4n-4}}(-(-q)^{4i-3}z^{-1})} & (k=n),\end{cases}$$

where $\Theta_p(z)$ is the theta function given by

$$\Theta_p(z)=(z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty,\quad (z;p)_\infty=\prod_{i=0}^{\infty}(1-p^iz),$$

and $[x]$ denotes the greatest integer not exceeding x. The explicit form of $\tau^{(k)}(z)$ is crucial to calculate the excitation spectra. From this formula, we can give explicit forms of the energy $\epsilon^{(k)}(\theta)$ and the momentum $p^{(k)}(\theta)$ with the rapidity variable θ,

$$e^{-ip^{(k)}(\theta)}=\tau^{(k)}(z),\quad \epsilon^{(k)}(\theta)=-(q-q^{-1})z\frac{d}{dz}\log\tau^{(k)}(z),\quad -z=e^{2\pi i\theta}.$$

To prove the commutation relations (2), it is enough to show that the vacuum expectation value of the both sides in (2) are equal (see section 5.1). Since the vacuum expectation values satisfy the q-KZ equation, we can obtain their explicit forms by finding appropriate solutions of the q-KZ equation (cf. [1]).

Here we state some comments about excitation spectra. By taking the scaling limit as in [8] and [5], we have the following relativistic spectrum:

$$\begin{aligned} P^{(k)}(\theta)&=2\mu\sin(\frac{\pi k}{2n-2})\,\mathrm{sh}(v), & E^{(k)}(\theta)&=2\mu\sin(\frac{\pi k}{2n-2})\,\mathrm{ch}(v), & &(1\le k\le n-2),\\ P^{(k)}(\theta)&=\mu\,\mathrm{sh}(v), & E^{(k)}(\theta)&=\mu\,\mathrm{ch}(v), & &(k=n-1,n),\end{aligned}$$

where $P^{(k)}(\theta)$, $E^{(k)}(\theta)$ and v can be considered as an appropriate scaled version of $p^{(k)}(\theta)$, $\epsilon^{(k)}(\theta)$ and θ respectively. These spectra are exactly same as those of the nonlinear sigma model in [10]. Furthermore, this spectrum coincides with the mass spectrum of the spin chain constructed from vector representation of $U_q(D_n^{(1)})$ in [5].

In general, the structure of the space of states turns out to be quite different when we change the region of the parameter q in the Hamiltonian $\mathcal{H}$, so we have to discuss the region where we can use the identification (1). We are not able to determine this region at this point,

and we only state a conjecture. The region where the identification (1) is effective is given by

$$-1 < q < 0.$$

These kind of conjectures are already given in [3], [5]. Let us consider the case of $n = 3$, then $U_q(D_3^{(1)})$ is isomorphic to $U_q(A_3^{(1)})$ and the spin representation of $U_q(D_3^{(1)})$ is the vector representation of $U_q(A_3^{(1)})$. The model related with the vector representation of $U_q(A_n^{(1)})$ is studied by Date and Okado in [3], and our conjecture coincides with theirs when $n = 3$. In this $U_q(A_n^{(1)})$-case, the validity of "quantum symmetry approach" on this region is supported by Bethe Ansatz results [8], however in our case, as far as the author knows, there is no similar Bethe Ansatz result to support the conjecture.

This paper is organized as follows. We prepare the notation and give the definition of the quantum affine algebra $U_q(D_n^{(1)})$ in Section 2. In the same section, we construct the spin representations of $U_q(D_n^{(1)})$ and its R-matrices. By using these R-matrices, the fusion representations are constructed in Section 3. We also give explicit forms of the R-matrices corresponding to the fusion representations there. In Section 4, we give vertex operators for the modules constructed in the previous sections and get two point functions. Commutation relations of vertex operators are obtained in Section 5. In the same section, we further make a comment on a relationship between excitation spectra and fusion procedures. In Section 6, we describe the formulation of the model. Finally, in Appendix, we give a detailed explanation on calculation of two point functions, and construct an isomorphism between the fusion representation and its dual.

1.1. **Acknowledgment.** The author would like to express his gratitude to Prof. M. Okado, Prof. K. Nagatomo and Prof. E. Date for their valuable advices. He also thanks to Prof. T. Nakanishi for his comment about relation between fusion and excitation spectra. Furthermore he expresses his thanks to Prof. M. Ikawa for giving him continuous encouragements.

2. Spin representations of the quantum group $U_q(D_n^{(1)})$

2.1. **Notation.** In this paper, we use the same notation in [11]. Let $\mathfrak{g}$ be the affine Lie algebra of type $D_n^{(1)}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition. Let α_i, $\alpha_i^\vee = h_i, (i = 0, 1, \dots, n)$ be the simple roots and the simple coroots and Λ_i be the fundamental weights i.e. $\langle \Lambda_i, h_j\rangle = \delta_{i,j}$. We denote the scaling element and the center by d and c then we can choose elements $h_1, h_2, \dots, h_n, c$ and d as a basis of

the Cartan subalgebra $\mathfrak{h}$. Let us define special elements $\rho, \delta \in \mathfrak{h}^*$ by $\rho = \sum_{i=0}^{n} \Lambda_i$ and $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. We also denote by $h^\vee$ the dual Coxeter number, in $D_n^{(1)}$-case, this is equal to $2n-2$. The invariant bilinear form is normalized by $(\alpha_i|\alpha_i) = 2$. Let $\mathring{\mathfrak{g}}$ be the Lie algebra of type D_n, underlying the affine Lie algebra $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$, we denote by $\bar{\lambda}$ the restriction to the finite part. We can write α_i, $\bar{\Lambda}_i$ and $\bar{\rho}$ by using orthonormal basis $\{\omega_1, \ldots, \omega_n\}$ of $\bar{\mathfrak{h}}^*$ (cf. [11]) as follows:

$$\alpha_i = \begin{cases} \omega_i - \omega_{i+1} & (1 \le i \le n-1), \\ \omega_{n-1} + \omega_n & (i = n), \end{cases} \tag{3}$$

$$\bar{\Lambda}_i = \begin{cases} \omega_1 + \cdots + \omega_i & (1 \le i \le n-2), \\ \frac{1}{2}(\omega_1 + \cdots + \omega_{n-1} - \omega_n) & (i = n-1), \\ \frac{1}{2}(\omega_1 + \cdots + \omega_{n-1} + \omega_n) & (i = n), \end{cases} \tag{4}$$

$$2\bar{\rho} = (2n-2)\omega_1 + (2n-4)\omega_2 + \cdots + 2\omega_{n-1}. \tag{5}$$

Let us denote $\hat{\sigma_1}$, $\hat{\sigma_2}$ and $\hat{\sigma_3}$ the following Dynkin diagram automorphism:

$$\begin{aligned} \hat{\sigma_1} &: i\text{-vertex} \longmapsto (1-i)\text{-vertex} && (i = 0, 1), \\ &: i\text{-vertex} \longmapsto i\text{-vertex} && (i = 2, 3, \cdots, n), \\ \hat{\sigma_2} &: i\text{-vertex} \longmapsto i\text{-vertex} && (i = 0, 1, \cdots, n-2), \\ &: i\text{-vertex} \longmapsto (2n-i-1)\text{-vertex} && (i = n-1, n), \\ \hat{\sigma_3} &: i\text{-vertex} \longmapsto (n-i)\text{-vertex} && (i = 0, 1, \cdots, n), \end{aligned}$$

and further we extend the action of these automorphisms to the weight lattice $P := \oplus_{i=1}^{n} \mathbb{Z}\Lambda_i$ by $\hat{\sigma_k}(\Lambda_i) = \Lambda_{\hat{\sigma_k}(i)}$. We also use the following notations:

$$\begin{aligned} [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, \\ \begin{bmatrix} i \\ j \end{bmatrix}_q &= \frac{[i]_q [i-1]_q \cdots [i+1-j]_q}{[j]_q [j-1]_q \cdots [1]_q}, \\ \xi &= q^{h^\vee} = q^{2n-2}, \\ p &= q^{2(h^\vee+1)} = q^{4n-2}. \end{aligned}$$

2.2. **Definition of the quantum group $U_q(D_n^{(1)})$.** The quantum group $U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{Q}(q)$ with generators e_i,

f_i, $t_i = q^{h_i}$ $(i = 0, 1, \cdots, n)$ and q^d. The defining relations are

$$\begin{aligned}
&t_i t_j = t_j t_i, \quad t_i q^d = q^d t_i,\\
&t_i e_j t_i^{-1} = q^{(\alpha_i|\alpha_j)} e_j, \quad t_i f_j t_i^{-1} = q^{-(\alpha_i|\alpha_j)} f_j,\\
&[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}},\\
&e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \qquad\qquad (\text{if } (\alpha_i|\alpha_j) = 0),\\
&\left.\begin{aligned} e_i(e_j)^2 - (q + q^{-1}) e_j e_i e_j + (e_j)^2 e_i = 0\\ f_i(f_j)^2 - (q + q^{-1}) f_j f_i f_j + (f_j)^2 f_i = 0 \end{aligned}\right. \quad (\text{if } (\alpha_i|\alpha_j) = -1).
\end{aligned}$$

This algebra $U_q(\mathfrak{g})$ has the following Hopf algebra structure:

$$\begin{aligned}
&\triangle(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \triangle(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,\\
&\triangle(t_i) = t_i \otimes t_i, \quad \triangle(q^d) = q^d \otimes q^d,\\
&a(e_i) = -t_i^{-1} e_i,\ a(f_i) = -f_i t_i,\ a(t_i) = t_i^{-1},\ a(q^d) = q^{-d}.
\end{aligned}$$

We denote by $U_q(\mathfrak{n}_+)$ (resp. $U_q(\mathfrak{n}_-)$) the subalgebras of $U_q(\mathfrak{g})$ which are generated by e_i $(i = 0, 1, \ldots, n)$ (resp. f_i $(i = 0, 1, \ldots, n)$) and also denote by $U'_q(\mathfrak{g})$ the subalgebra generated by e_i, f_i, t_i $(i = 0, 1, \ldots, n)$. For any $U'_q(\mathfrak{g})$-module (π, V), we can define $U_q(\mathfrak{g})$-module structure (π_z, V_z) by

$$\begin{aligned}
&V_z = \mathbb{Q}(q)[z, z^{-1}] \otimes V,\\
&\quad \pi_z(e_i)(z^n \otimes v) = z^{n+\delta_{i,0}} \otimes \pi(e_i) v,\\
&\quad \pi_z(f_i)(z^n \otimes v) = z^{n-\delta_{i,0}} \otimes \pi(f_i) v,\\
&\quad \pi_z(t_i)(z^n \otimes v) = z^n \otimes \pi(t_i) v,\\
&\quad \pi_z(q^d)(z^n \otimes v) = (qz)^n \otimes v,
\end{aligned} \tag{6}$$

and this $U_q(\mathfrak{g})$-module (π_z, V_z) is called the affinization of (π, V).

The Dynkin diagram automorphisms can be extended to algebra automorphisms of $U'_q(\mathfrak{g})$ as follows:

$$\begin{aligned}
&\hat{\sigma_1}(x_i) = x_{1-i} (i = 0, 1), && \hat{\sigma_1}(x_i) = x_i \quad (i = 2, 3, \cdots, n),\\
&\hat{\sigma_2}(x_i) = x_i \quad (i = 0, 1, \cdots, n-2), && \hat{\sigma_2}(x_i) = x_{2n-i-1} (i = n-1, n),\\
&\hat{\sigma_3}(x_i) = x_{n-i} (i = 0, 1, \cdots, n),
\end{aligned}$$

where x_i stands for e_i, f_i or t_i.

2.3. **R-matrices.** Let (π^{W_1}, W_1) and (π^{W_2}, W_2) be finite-dimensional $U_q'(\mathfrak{g})$-modules. An operator $R(z_1/z_2) \in \mathrm{End}(W_1 \otimes W_2)$ is called R-matrix if it has the following intertwining property:

$$\begin{aligned} &R^{W_1,W_2}(z_1/z_2)(\pi^{W_1}_{z_1} \otimes \pi^{W_2}_{z_2}) \triangle (x) \\ &\quad = (\pi^{W_1}_{z_1} \otimes \pi^{W_2}_{z_2}) \triangle' (x) R^{W_1,W_2}(z_1/z_2) \quad (x \in U_q(\mathfrak{g})), \end{aligned} \tag{7}$$

where $\triangle' = P \circ \triangle$.

Let (π^{W_3}, W_3) be the third representation of $U_q'(\mathfrak{g})$. The R-matrices satisfy the Yang-Baxter equation on $W_1 \otimes W_2 \otimes W_3$,

$$\begin{aligned} &R^{W_1,W_2}(z_1/z_2) R^{W_1,W_3}(z_1/z_3) R^{W_2,W_3}(z_2/z_3) \\ &\quad = R^{W_2,W_3}(z_2/z_3) R^{W_1,W_3}(z_1/z_3) R^{W_1,W_2}(z_1/z_2), \end{aligned} \tag{8}$$

where R^{W_i,W_j} acts non-trivially on the i-th and the j-th components.

The modified universal R-matrix $\mathcal{R}'(z)$ is defined by

$$\mathcal{R}'(z) = q^{-\sum_{i=1}^n h_i \otimes \bar{\Lambda}_i} \sum_{\beta,i} z^{<d,\beta>} u_{\beta,i} \otimes u^i_{-\beta} \ (\in U_q'(\mathfrak{g}) \hat{\otimes} \mathfrak{U}_\mathfrak{q}'(\mathfrak{g})), \tag{9}$$

where $\{u_{\beta,i}\}$ (resp. $\{u^i_{-\beta}\}$) are dual bases of weight β (resp. $-\beta$) component of the subalgebra $U_q(\mathfrak{n}_+)$ (resp. $U_q(\mathfrak{n}_-)$) and $\hat{\otimes}$ means completion in formal topology with respect to the weight decomposition of $U_q(\mathfrak{n}_+)$ (cf. [2]). In the same way as [2], we can determine the image of the modified universal R matrix under the representation

$$\pi^{W_1} \otimes \pi^{W_2} \ : \ U_q'(\mathfrak{g}) \hat{\otimes} \mathfrak{U}_\mathfrak{q}'(\mathfrak{g}) \longrightarrow \mathrm{End}(\mathfrak{W}_1 \otimes \mathfrak{W}_2).$$

If our R-matrix on $W_1 \otimes W_2$ is uniquely determined up to multiple scalar factor then the image of $\mathcal{R}'(z)$ has the following form:

$$(\pi^{W_1} \otimes \pi^{W_2})(\mathcal{R}'(z)) = \beta^{W_1,W_2}(z) R^{W_1,W_2}(z). \tag{10}$$

To calculate the scalar factor $\beta^{W_1,W_2}(z)$, we have to obtain the explicit form of a scalar function $\alpha^{W_1,W_2}(z)$ appeared in the second inversion relation:

$$\alpha^{W_1,W_2}(z)(((R^{W_1,W_2}(z)^{-1})^{t_1})^{-1})^{t_1} = (q^{-2\rho} \otimes 1) R^{W_1,W_2}(\xi^{-2} z)(q^{2\rho} \otimes 1), \tag{11}$$

where the symbol t_1 means the transpose in the first component. It is known that the scalar function $\alpha^{W_1,W_2}(z)$ for the image of $\mathcal{R}'(z)$ is equal to one i.e.

$$((((\beta^{W_1,W_2}(z) R^{W_1,W_2}(z))^{-1})^{t_1})^{-1})^{t_1} = (q^{-2\rho} \otimes 1) \beta^{W_1,W_2}(\xi^{-2} z) R^{W_1,W_2}(\xi^{-2} z)(q^{2\rho} \otimes 1).$$

Comparing this equation with (11), we have

$$\alpha^{W_1,W_2}(z) \beta^{W_1,W_2}(z)^{-1} = \beta^{W_1,W_2}(\xi^{-2} z)^{-1}.$$

We can determine the explicit form of $\beta^{W_1,W_2}(z)$ as a solution of this difference equation. In the case related with the spin representation, more detailed explanation on calculation of $\alpha^{W_1,W_2}(z)$ and $\beta^{W_1,W_2}(z)$ is given in Appendix.

2.4. **Spin representations.** Let $V_{1/2}$ be a 2-dimensional vector space spanned by vectors $v_{1/2}$ and $v_{-1/2}$ over the field $\mathbb{Q}(q^{1/2})$. We define operators X^+, X^- and T acting on $V_{1/2}$ by

$$X^+ v_\gamma = v_{\gamma+1},\ X^- v_\gamma = v_{\gamma-1},\ T v_\gamma = q^\gamma v_\gamma,$$

where if $\gamma \neq \pm 1/2$ then $v_\gamma = 0$ i.e. matrices corresponding to X^+, X^- and T are given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}.$$

By using these operators, we can define a representation of $U'_q(\mathfrak{g})$ on a vector space $V^{(sp)} = (V_{1/2})^{\otimes n}$ as follows [6]:

$$\begin{aligned}
&\pi^{(sp)}(e_0) = X^- \otimes X^- \otimes 1 \otimes \cdots \otimes 1, \\
&\pi^{(sp)}(t_0) = T^{-1} \otimes T^{-1} \otimes 1 \otimes \cdots \otimes 1, \\
&\pi^{(sp)}(e_i) = 1 \otimes \cdots \otimes 1 \otimes \overset{i\text{-th}}{X^+} \otimes \overset{(i+1)\text{-th}}{X^-} \otimes 1 \otimes \cdots \otimes 1 \quad (1 \leq i \leq n-1), \\
&\pi^{(sp)}(t_i) = 1 \otimes \cdots \otimes 1 \otimes \overset{i\text{-th}}{T} \otimes \overset{(i+1)\text{-th}}{T^{-1}} \otimes 1 \otimes \cdots \otimes 1 \quad (1 \leq i \leq n-1), \\
&\pi^{(sp)}(e_n) = 1 \otimes \cdots \otimes 1 \otimes X^+ \otimes X^+, \\
&\pi^{(sp)}(t_n) = 1 \otimes \cdots \otimes 1 \otimes T \otimes T, \\
&\pi^{(sp)}(f_i) = \pi^{(sp)}(e_i)^t.
\end{aligned} \tag{12}$$

We denote a vector $v_{\gamma_1} \otimes \cdots \otimes v_{\gamma_n} \in V^{(sp)}$ by $v_{(\varepsilon_1,\ldots,\varepsilon_n)}$ $(\varepsilon_k = \mathrm{sgn}(\gamma_k))$. The weight of the element $v_{(\varepsilon_1,\ldots,\varepsilon_n)}$ is given by $\sum_{i=1}^n \varepsilon_i \omega_i/2$, where $\{\omega_k\}$ is the orthogonal basis of $\bar{\mathfrak{h}}^*$. Let $V^{(+)}$ and $V^{(-)}$ be the subspaces spanned by $\{v_{(\varepsilon_1,\ldots,\varepsilon_n)} | \prod_{i=1}^n \varepsilon_i = +\}$ and $\{v_{(\varepsilon_1,\ldots,\varepsilon_n)} | \prod_{i=1}^n \varepsilon_i = -\}$, then we can easily show that they are irreducible submodules. These irreducible representations are called spin representations and are denoted by $(\pi^{(\pm)}, V^{(\pm)})$.

There exists isomorphisms of $U_q(\mathfrak{g})$-module (cf. [5])

$$C_\pm^{(sp)} \ : \ V_{z\xi^{\mp}}^{(sp)} \longrightarrow (V_z^{(sp)})^{*a^{\pm 1}}. \tag{13}$$

We denote the restrictions of $C_{\pm}^{(sp)}$ to the irreducible components $V^{(+)}$ (resp. $V^{(-)}$) by $C_{\pm}^{(+)}$ (resp. $C_{\pm}^{(-)}$) i.e.

$$\begin{aligned} C_{\pm}^{(+)} &: V_{z\xi^{\mp}}^{(+)} \longrightarrow (V_z^{(\varepsilon)})^{*a^{\pm 1}}, \\ C_{\pm}^{(-)} &: V_{z\xi^{\mp}}^{(-)} \longrightarrow (V_z^{(-\varepsilon)})^{*a^{\pm 1}}, \end{aligned} \tag{14}$$

where ε is given by

$$\varepsilon = \begin{cases} + & (n : \text{even}), \\ - & (n : \text{odd}). \end{cases}$$

We can define the action of the Dynkin diagram automorphisms $\hat{\sigma_1}$, $\hat{\sigma_2}$ and $\hat{\sigma_3}$ to the spin representations as follows:

$$\begin{aligned} \hat{\sigma_1}(v_{(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)}) &= v_{(-\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)}, \\ \hat{\sigma_2}(v_{(\varepsilon_1,\cdots,\varepsilon_{n-1},\varepsilon_n)}) &= v_{(\varepsilon_1,\cdots,\varepsilon_{n-1},-\varepsilon_n)}, \\ \hat{\sigma_3}(v_{(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)}) &= v_{(-\varepsilon_n,\cdots,-\varepsilon_2,-\varepsilon_1)}. \end{aligned} \tag{15}$$

2.5. **R-matrices related with the spin representations.** In this subsection, we describe recursive forms of the R-matrices related with the spin representations. In [6], the R-matrices related with the spin representations of $U_q(B_n^{(1)})$ and $U_q(D_n^{(1)})$ are determined, and the R-matrix for $U_q(B_n^{(1)})$ is recursively expressed by using the one for $U_q(B_{n-1}^{(1)})$. We give a similar recursive relation in the case of $U_q(D_n^{(1)})$.

We normalize the R-matrix $R^{V^{(\varepsilon_1)},V^{(\varepsilon_2)}}(z)$ by

$$R^{V^{(\varepsilon_1)},V^{(\varepsilon_2)}}(z) v_{(+,+,\dots,+,\varepsilon_1)} \otimes v_{(+,+,\dots,+,\varepsilon_2)} = v_{(+,+,\dots,+,\varepsilon_1)} \otimes v_{(+,+,\dots,+,\varepsilon_2)},$$

and denote this R-matrix by $\bar{R}_n^{(\varepsilon_1,\varepsilon_2)}(z)$ where n stands for the rank of $\bar{\mathfrak{g}}$. Expressing the rank of $\mathfrak{g}$ clearly, we denote the spin representations $V^{(\pm)}$ by $V_n^{(\pm)}$ only in this subsection. In order to describe recursive formulae of R-matrices, we consider an isomorphism of vector spaces

$$V_n^{(\varepsilon_1)} \otimes V_n^{(\varepsilon_2)} \longrightarrow \sum_{\substack{\eta_1,\eta_2=\pm \\ \eta_1'=\varepsilon_1\eta_1 \\ \eta_2'=\varepsilon_2\eta_2}} \mathbb{Q}(q) v_{\eta_1,\eta_2} \otimes (V_{n-1}^{(\eta_1')} \otimes V_{n-1}^{(\eta_2')}), \tag{16}$$

which maps

$$v_{(\nu_1,\nu_2\dots,\nu_n)} \otimes v_{(\mu_1,\mu_2\dots,\mu_n)} \;\mapsto\; v_{\nu_1,\mu_1} \otimes \big(v_{(\nu_2\dots,\nu_n)} \otimes v_{(\mu_2\dots,\mu_n)}\big),$$

where we denote $v_{\nu_1} \otimes v_{\mu_1}$ by v_{ν_1,μ_1}. We put

$$a(z) = \frac{q(1-z)}{1-q^2 z}, \quad b(z) = \frac{1-q^2}{1-q^2 z}, \quad c(z) = \frac{b(z)}{a(z)}. \tag{17}$$

By using this isomorphism, we have the following expressions of R-matrices:

for $n \geq 2$

$$
\begin{aligned}
&\bar{R}_n^{(\varepsilon,\varepsilon)}(z) v_{++} \otimes u_{n-1} = v_{++} \otimes \bar{R}_{n-1}^{(\varepsilon,\varepsilon)}(z) u_{n-1}, \\
&\bar{R}_n^{(\varepsilon,\varepsilon)}(z) v_{+-} \otimes u_{n-1} \\
&\qquad = v_{+-} \otimes a(z) \bar{R}_{n-1}^{(\varepsilon,-\varepsilon)}(q^2 z) u_{n-1} + v_{-+} \otimes z b(z) \bar{R}_{n-1}^{(-\varepsilon,\varepsilon)}(q^2 z) \sigma_{n-1} X_{n-1}^{(\varepsilon,-\varepsilon)} u_{n-1}, \\
&\bar{R}_n^{(\varepsilon,\varepsilon)}(z) v_{-+} \otimes u_{n-1} \\
&\qquad = v_{+-} \otimes b(z) \bar{R}_{n-1}^{(\varepsilon,-\varepsilon)}(q^2 z) \sigma_{n-1} X_{n-1}^{(-\varepsilon,\varepsilon)} u_{n-1} + v_{-+} \otimes a(z) \bar{R}_{n-1}^{(-\varepsilon,\varepsilon)}(q^2 z) u_{n-1}, \\
&\bar{R}_n^{(\varepsilon,\varepsilon)}(z) v_{--} \otimes u_{n-1} = v_{--} \otimes \bar{R}_{n-1}^{(-\varepsilon,-\varepsilon)}(z) u_{n-1},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\bar{R}_n^{(\varepsilon,-\varepsilon)}(z) v_{++} \otimes u_{n-1} = v_{++} \otimes \bar{R}_{n-1}^{(\varepsilon,-\varepsilon)}(z) u_{n-1}, \\
&\bar{R}_n^{(\varepsilon,-\varepsilon)}(z) v_{+-} \otimes u_{n-1} \\
&\qquad = v_{+-} \otimes \bar{R}_{n-1}^{(\varepsilon,\varepsilon)}(q^2 z) u_{n-1} + v_{-+} \otimes z c(z) \bar{R}_{n-1}^{(-\varepsilon,-\varepsilon)}(q^2 z) \sigma_{n-1} X_{n-1}^{(\varepsilon,\varepsilon)} u_{n-1}, \\
&\bar{R}_n^{(\varepsilon,-\varepsilon)}(z) v_{-+} \otimes u_{n-1} \\
&\qquad = v_{+-} \otimes c(z) \bar{R}_{n-1}^{(\varepsilon,\varepsilon)}(q^2 z) \sigma_{n-1} X_{n-1}^{(-\varepsilon,-\varepsilon)} u_{n-1} + v_{-+} \otimes \bar{R}_{n-1}^{(-\varepsilon,-\varepsilon)}(q^2 z) u_{n-1}, \\
&\bar{R}_n^{(\varepsilon,-\varepsilon)}(z) v_{--} \otimes u_{n-1} = v_{--} \otimes \bar{R}_{n-1}^{(-\varepsilon,\varepsilon)}(z) u_{n-1},
\end{aligned}
\tag{19}
$$

for $n = 1$

$$
\bar{R}_1^{(\varepsilon,\varepsilon)}(z) = 1, \quad \bar{R}_1^{(\varepsilon,-\varepsilon)}(z) = 1,
$$

where an involution σ_n is defined by

$$
\sigma_n(v_{(\varepsilon_1,\cdots,\varepsilon_n)} \otimes v_{(\eta_1,\cdots,\eta_n)}) = v_{(\varepsilon_1,\cdots,\varepsilon_{n-1},-\varepsilon_n)} \otimes v_{(\eta_1,\cdots,\eta_{n-1},-\eta_n)},
$$

and $X_n^{(\varepsilon_1,\varepsilon_2)}$ is a linear operator on $V_n^{(\varepsilon_1)} \otimes V_n^{(\varepsilon_2)}$ that is defined inductively as follows:

for $n \geq 2$

$$
\begin{aligned}
&X_n^{(\varepsilon_1,\varepsilon_2)} v_{++} \otimes u_{n-1} = v_{++} \otimes X_{n-1}^{(\varepsilon_1,\varepsilon_2)} u_{n-1} \\
&X_n^{(\varepsilon_1,\varepsilon_2)} v_{+-} \otimes u_{n-1} = -q^{-1} v_{+-} \otimes X_{n-1}^{(\varepsilon_1,-\varepsilon_2)} u_{n-1} + v_{-+} \otimes \sigma_{n-1} u_{n-1}, \\
&X_n^{(\varepsilon_1,\varepsilon_2)} v_{-+} \otimes u_{n-1} = v_{+-} \otimes \sigma_{n-1} u_{n-1} - q v_{-+} \otimes X_{n-1}^{(-\varepsilon_1,\varepsilon_2)} u_{n-1}, \\
&X_n^{(\varepsilon_1,\varepsilon_2)} v_{--} \otimes u_{n-1} = v_{--} \otimes X_{n-1}^{(-\varepsilon_1,-\varepsilon_2)} u_{n-1},
\end{aligned}
\tag{20}
$$

for $n = 1$

$$
X_1^{(\varepsilon,\varepsilon)} = 0, \qquad X_1^{(\varepsilon,-\varepsilon)} = 1.
$$

We denote by $\alpha_n^{(\varepsilon_1,\varepsilon_2)}(z)$ the scalar function $\alpha^{V^{(\varepsilon_1)},V^{(\varepsilon_2)}}(z)$ in (11). Then we can find

$$\alpha_n^{(\varepsilon_1,\varepsilon_2)}(z)=\begin{cases}\displaystyle\prod_{i=1}^{[\frac{n}{2}]}\frac{(1-q^{-4i+4}z)(1-q^{-4n+4i}z)}{(1-q^{-4i+2}z)(1-q^{-4n+4i+2}z)} & (\varepsilon_1=\varepsilon_2),\\[2ex] \displaystyle\prod_{i=1}^{[\frac{n-1}{2}]}\frac{(1-q^{-4i+2}z)(1-q^{-4n+4i+2}z)}{(1-q^{-4i}z)(1-q^{-4n+4i+4}z)} & (\varepsilon_1\neq\varepsilon_2).\end{cases} \tag{21}$$

This fact is proved in Appendix A.

We also denote by $\beta_n^{(\varepsilon_1,\varepsilon_2)}(z)$ the scalar factor $\beta^{V^{(\varepsilon_1)},V^{(\varepsilon_2)}}(z)$ in (10). By using the explicit form of the function $\alpha_n^{(\varepsilon_1,\varepsilon_2)}(z)$, we can show (see Appendix B)

$$\beta_n^{(\varepsilon_1,\varepsilon_2)}(z)=\begin{cases}\displaystyle q^{-\frac{n}{4}}\prod_{i=1}^{[\frac{n}{2}]}\frac{(q^{4n-4i-2}z;\xi^2)_\infty(q^{4i-2}z;\xi^2)_\infty}{(q^{4n-4i}z;\xi^2)_\infty(q^{4i-4}z;\xi^2)_\infty} & (\varepsilon_1=\varepsilon_2),\\[2ex] \displaystyle q^{-\frac{n}{4}+\frac{1}{2}}\prod_{i=1}^{[\frac{n-1}{2}]}\frac{(q^{4n-4i-4}z;\xi^2)_\infty(q^{4i}z;\xi^2)_\infty}{(q^{4n-4i-2}z;\xi^2)_\infty(q^{4i-2}z;\xi^2)_\infty} & (\varepsilon_1\neq\varepsilon_2).\end{cases} \tag{22}$$

3. Fusion construction

We construct the fusion representations and R-matrices related with them.

3.1. **Construction of $V^{(k)}$.**

Since $V^{(+)}$ and $V^{(-)}$ are the highest weight modules corresponding to the highest weights $\bar{\Lambda}_n$ and $\bar{\Lambda}_{n-1}$ as $U_q(\mathring{\mathfrak{g}})$-modules, we denote $V^{(+)}$ and $V^{(-)}$ by $V^{(n)}$ and $V^{(n-1)}$. We also denote $\bar{R}_n^{(+,+)}(z)$ and $\bar{R}_n^{(+,-)}(z)$ by $\bar{R}^{(n,n)}(z)$ and $\bar{R}^{(n,n-1)}(z)$ respectively.

The R-matrix $\bar{R}^{(n,n)}(z)$ (resp. $\bar{R}^{(n,n-1)}(z)$) have a pole of order one at $z=q^{-2n+2k+2}$ ($k\equiv n \pmod 2$) (resp. $z=q^{-2n+2k+2}$ ($k\not\equiv n \pmod 2$)) for all $k=1,2,\dots,n-2$. We eliminate these poles by multiplying an appropriate scalar factor to the R-matrix as follows.

$$R^{(n,n)}(z)=\prod_{i=1}^{[\frac{n}{2}]}(1-q^{4i-2}z)\bar{R}^{(n,n)}(z),$$
$$R^{(n,n-1)}(z)=\prod_{i=1}^{[\frac{n-1}{2}]}(1-q^{4i}z)\bar{R}^{(n,n-1)}(z).$$

By using these R-matrices, let us define operators $T^{(k)}$ $(1 \le k \le n-2)$ by

$$T^{(k)} = \begin{cases} R^{(n,n)}(q^{-2n+2k+2}) & \in \mathrm{End}(V^{(n)} \otimes V^{(n)}) \qquad (k \equiv n \pmod 2), \\ R^{(n,n-1)}(q^{-2n+2k+2}) & \in \mathrm{End}(V^{(n)} \otimes V^{(n-1)}) \qquad (k \not\equiv n \pmod 2). \end{cases}$$

The following function $\varphi^{(n)} : \mathbb{N} \to \{n-1, n\}$ is useful in this section:

$$\varphi^{(n)}(i) = \begin{cases} n & (i : \text{even}), \\ n-1 & (i : \text{odd}). \end{cases} \tag{23}$$

We introduce a vector space

$$V^{(k)} = (V^{(n)} \otimes V^{(n')}) / \ker T^{(k)},$$

where $1 \le k \le n-2$ and $n' = \varphi^{(n)}(n-k)$. We define an action of $U_q'(\mathfrak{g})$

$$\tilde{\pi}_z^{(k)} \ : \ U_q'(\mathfrak{g}) \longrightarrow \mathrm{End}(\mathfrak{V}^{(\mathfrak{n})} \otimes \mathfrak{V}^{(\mathfrak{n}')}),$$

by

$$\tilde{\pi}_z^{(k)}(x) = (\pi^{(n)}_{(-q)^{-n+k+1}z} \otimes \pi^{(n')}_{(-q)^{n-k-1}z}) \circ \triangle(x),$$

then we have the following proposition.

Proposition 3.1. *For all $x \in U_q'(\mathfrak{g})$,*

$$\tilde{\pi}_z^{(k)}(x) \ker T^{(k)} \subset \ker T^{(k)}. \tag{24}$$

Then $\tilde{\pi}_z^{(k)}$ induces a representation of $U_q'(\mathfrak{g})$ on $V^{(k)}$.

Proof. Let v be an arbitrary element in $\ker T^{(k)}$. By means of the intertwiner property of the R-matrices in (7), we have

$$\begin{aligned} T^{(k)}\tilde{\pi}_z^{(k)}(x)v &= R^{(n,n')}(q^{-2n+2k+2})(\pi^{(n)}_{(-q)^{-n+k+1}z} \otimes \pi^{(n')}_{(-q)^{n-k-1}z}) \triangle (x)v \\ &= (\pi^{(n)}_{(-q)^{-n+k+1}z} \otimes \pi^{(n')}_{(-q)^{n-k-1}z}) \triangle' (x) R^{(n,n')}(q^{-2n+2k+2})v \\ &= 0, \end{aligned}$$

for all $x \in U_q'(\mathfrak{g})$. Then $\tilde{\pi}_z^{(k)}(x)v \in \ker T^{(k)}$. Q.E.D

We define an action of q^d on $V^{(k)}$ which is a representation of $U_q'(\mathfrak{g})$ induced by the above $\tilde{\pi}_z^{(k)}$ as similar to (6) and denote it by $(\pi_z^{(k)}, V_z^{(k)})$. The following isomorphisms of $U_q(\mathfrak{g})$-module are known in [5]:

$$C_\pm^{(k)} \ : \ V^{(k)}_{z\xi^\mp} \longrightarrow (V_z^{(k)})^{*a^{\pm 1}} \quad (k = 1, 2, \cdots, n-2). \tag{25}$$

Explicit forms of $C_\pm^{(k)}$ are given in Appendix.

3.2. **R-matrices related with $V^{(k)}$.** For $k = 1, 2, \ldots, n-2$, we explicitly construct R-matrices on $V^{(k)} \otimes V^{(n)}$, $V^{(k)} \otimes V^{(n-1)}$, $V^{(n)} \otimes V^{(k)}$ and $V^{(n-1)} \otimes V^{(k)}$ (see (7)). Let m be n or $n-1$. We define operators

$$R^{(m,k)}(z) \in \mathrm{End}(V^{(m)} \otimes V^{(n)} \otimes V^{(n')}),$$
$$R^{(k,m)}(z) \in \mathrm{End}(V^{(n)} \otimes V^{(n')} \otimes V^{(m)}),$$

by

$$R^{(m,k)}(z) = \bar{R}^{(m,n')}((-q)^{-n+k+1}z)_{13}\bar{R}^{(m,n)}((-q)^{n-k-1}z)_{12},$$
$$R^{(k,m)}(z) = \bar{R}^{(n,m)}((-q)^{-n+k+1}z)_{13}\bar{R}^{(n',m)}((-q)^{n-k-1}z)_{23},$$

where $n' = \varphi^{(n)}(n-k)$ and the subscripts of these R-matrices indicate the components that each operator acts on, that is, R_{ij} non-trivially acts on the i-th and the j-th components.

Here we will show that

$$R^{(m,k)}(z)(V^{(m)} \otimes \ker T^{(k)}) \subset V^{(m)} \otimes \ker T^{(k)},$$

and so $R^{(m,k)}(z)$ (resp. $R^{(k,m)}(z)$) define operators on $V^{(m)} \otimes V^{(k)}$ (resp. $V^{(k)} \otimes V^{(m)}$). In fact, for arbitrary $v \in V^{(m)} \otimes \ker T^{(k)}$, by using the Yang-Baxter equation (8), we have

$$\begin{aligned}
&(\mathrm{id} \otimes T^{(k)})R^{(m,k)}(z)v \\
&= R^{(n,n')}(q^{-2n+2k+2})_{23}\bar{R}^{(m,n')}((-q)^{-n+k+1}z)_{13}\bar{R}^{(m,n)}((-q)^{n-k-1}z)_{12}v \\
&= \bar{R}^{(m,n)}((-q)^{n-k-1}z)_{12}\bar{R}^{(m,n')}((-q)^{-n+k+1}z)_{13}R^{(n,n')}(q^{-2n+2k+2})_{23}v \\
&= 0.
\end{aligned}$$

Hence $(\mathrm{id} \otimes T^{(k)})R^{(m,k)}(z)v \in V^{(m)} \otimes \ker T^{(k)}$. We can prove

$$R^{(k,m)}(z)(\ker T^{(k)} \otimes V^{(m)}) \subset \ker T^{(k)} \otimes V^{(m)}$$

similarly. Then we have

Proposition 3.2. *$R^{(m,k)}(z)$ and $R^{(k,m)}(z)$ are well defined as operators acting on $V^{(m)} \otimes V^{(k)}$ and $V^{(k)} \otimes V^{(m)}$ respectively.*

Operators $R^{(m,k)}(z)$ and $R^{(k,m)}(z)$ satisfy the intertwining property (7) i.e. they are R-matrices on $V^{(k)} \otimes V^{(m)}$ and $V^{(m)} \otimes V^{(k)}$.

Proposition 3.3. *For all $x \in U_q(\mathfrak{g})$,*

$$R^{(m,k)}(z_1/z_2)(\pi^{(m)}_{z_1} \otimes \pi^{(k)}_{z_2}) \triangle (x) = (\pi^{(m)}_{z_1} \otimes \pi^{(k)}_{z_2}) \triangle' (x) R^{(m,k)}(z_1/z_2), \quad (26)$$
$$R^{(k,m)}(z_1/z_2)(\pi^{(k)}_{z_1} \otimes \pi^{(m)}_{z_2}) \triangle (x) = (\pi^{(k)}_{z_1} \otimes \pi^{(m)}_{z_2}) \triangle' (x) R^{(k,m)}(z_1/z_2). \quad (27)$$

Proof. Here we only prove (26). We denote

$$\triangle(x)=\sum x_{(1)}\otimes x_{(2)},$$

$$(\triangle\otimes \mathrm{id})(x)=(\mathrm{id}\otimes\triangle)(x)=\sum x_{(1)}\otimes x_{(2)}\otimes x_{(3)}.$$

For all $x\in U_q(\mathfrak{g})$,

$$\begin{aligned}
&R^{(m,k)}(z_1/z_2)(\pi^{(m)}_{z_1}\otimes\pi^{(k)}_{z_2})\triangle(x)\\
&\quad=R^{(m,k)}(z_1/z_2)\{\textstyle\sum\pi^{(m)}_{z_1}(x_{(1)})\otimes\pi^{(k)}_{z_2}(x_{(2)})\}\\
&\quad=\bar{R}^{(m,n')}((-q)^{-n+k+1}z_1/z_2)_{13}\bar{R}^{(m,n)}((-q)^{n-k-1}z_1/z_2)_{12}\\
&\qquad\times\{\textstyle\sum\pi^{(m)}_{z_1}(x_{(1)})\otimes\pi^{(n)}_{(-q)^{-n+k+1}z_2}(x_{(2)})\otimes\pi^{(n')}_{(-q)^{n-k-1}z_2}(x_{(3)})\}\\
&\quad=\bar{R}^{(m,n')}((-q)^{-n+k+1}z_1/z_2)_{13}\{\textstyle\sum\pi^{(m)}_{z_1}(x_{(2)})\otimes\pi^{(n)}_{(-q)^{-n+k+1}z_2}(x_{(1)})\otimes\pi^{(n')}_{(-q)^{n-k-1}z_2}(x_{(3)})\}\\
&\qquad\times\bar{R}^{(m,n)}((-q)^{n-k-1}z_1/z_2)_{12}\\
&\quad=\{\textstyle\sum\pi^{(m)}_{z_1}(x_{(3)})\otimes\pi^{(n)}_{(-q)^{-n+k+1}z_2}(x_{(2)})\otimes\pi^{(n')}_{(-q)^{n-k-1}z_2}(x_{(1)})\}\\
&\qquad\times\bar{R}^{(m,n')}((-q)^{-n+k+1}z_1/z_2)_{13}\bar{R}^{(m,n)}((-q)^{n-k-1}z_1/z_2)_{12}\\
&\quad=\{\textstyle\sum\pi^{(m)}_{z_1}(x_{(2)})\otimes\pi^{(k)}_{z_2}(x_{(1)})\}R^{(m,k)}(z_1/z_2)\\
&\quad=(\pi^{(m)}_{z_1}\otimes\pi^{(k)}_{z_2})\triangle'(x)R^{(m,k)}(z_1/z_2).
\end{aligned}$$

Q.E.D

For our aim of finding commutation relations of vertex operators, it is enough to consider R-matrices in the cases listed below (see Section 4.3):

I	$V^{(k)}\otimes V^{(n)}$	n:even	k:even
II	$V^{(k)}\otimes V^{(n)}$	n:odd	k:even
III	$V^{(k)}\otimes V^{(n-1)}$	n:even	k:odd
IV	$V^{(k)}\otimes V^{(n-1)}$	n:odd	k:odd
V	$V^{(n)}\otimes V^{(k)}$	n:even	k:even
VI	$V^{(n)}\otimes V^{(k)}$	n:odd	k:even
VII	$V^{(n-1)}\otimes V^{(k)}$	n:even	k:odd
VIII	$V^{(n-1)}\otimes V^{(k)}$	n:odd	k:odd

where $1\le k\le n-2$ in all these cases.

Let $u^{(k)}$ be the highest weight vector of the fundamental representation $V^{(k)}$. We normalize R-matrices such that the eigenvalues of elements $u^{(k)}\otimes u^{(m)}$ (resp. $u^{(m)}\otimes u^{(k)}$) for $\bar{R}^{(k,m)}_n(z)$ (resp. $\bar{R}^{(m,k)}_n(z)$) are equal to one, then we have

I and II $$\bar{R}_n^{(k,n)}(z)=\prod_{i=1}^{[\frac{n-k}{2}]} a(q^{4i-4}(-q)^{-n+k+1}z)^{-1}R_n^{(k,n)}(z),$$

III and IV $$\bar{R}_n^{(k,n-1)}(z)=\prod_{i=1}^{[\frac{n-k-1}{2}]} a(q^{4i-2}(-q)^{-n+k+1}z)^{-1}R_n^{(k,n-1)}(z),$$

V and VII $$\bar{R}_n^{(m,k)}(z)=\prod_{i=1}^{[\frac{n-k}{2}]} a(q^{4i-4}(-q)^{-n+k+1}z)^{-1}R_n^{(m,k)}(z),$$

VI and VIII $$\bar{R}_n^{(m,k)}(z)=\prod_{i=1}^{[\frac{n-k-1}{2}]} a(q^{4i-2}(-q)^{-n+k+1}z)^{-1}R_n^{(m,k)}(z),$$

where $m=\varphi^{(n)}(k)$ and $a(z)=q(1-z)/(1-q^2z)$.

We denote the scalar functions $\alpha^{V^{(k)},V^{(m)}}(z)$, $\alpha^{V^{(m)},V^{(k)}}(z)$, $\beta^{V^{(k)},V^{(m)}}(z)$ and $\beta^{V^{(m)},V^{(k)}}(z)$ by $\alpha_n^{(k,m)}(z)$, $\alpha_n^{(m,k)}(z)$, $\beta_n^{(k,m)}(z)$ and $\beta_n^{(m,k)}(z)$. In a similar way to Section 2.6, we can show

$$\alpha_n^{(k,m)}(z)=\alpha_n^{(m,k)}(z)=\frac{(1+(-1)^{n-k}q^k\xi^{-1/2}z)(1+(-1)^{n-k}q^{-k}\xi^{-3/2}z)}{(1+(-1)^{n-k}q^{-k}\xi^{-1/2}z)(1+(-1)^{n-k}q^{k}\xi^{-3/2}z)},$$

$$\beta_n^{(k,m)}(z)=\beta_n^{(m,k)}(z)=q^{-\frac{k}{2}}\frac{(-(-1)^{n-k}q^k\xi^{1/2}z;\xi^2)_\infty(-(-1)^{n-k}q^{-k}\xi^{3/2}z;\xi^2)_\infty}{(-(-1)^{n-k}q^{-k}\xi^{1/2}z;\xi^2)_\infty(-(-1)^{n-k}q^{k}\xi^{3/2}z;\xi^2)_\infty}.$$

4. Vertex operators and two point functions

In this section we define vertex operators and calculate two point functions.

4.1. **Vertex operators.** Vertex operators are the following homomorphisms of $U_q(\mathfrak{g})$-modules:

$$\text{(type I)}\ \ \tilde{\Phi}_\lambda^{\mu\,V^{(k)}}(z)\ :\ V(\lambda)\longrightarrow \hat{V}(\mu)\otimes V_z^{(k)},$$
$$\text{(type II)}\ \ \tilde{\Phi}_\lambda^{V^{(k)}\,\mu}(z)\ :\ V(\lambda)\longrightarrow V_z^{(k)}\otimes \hat{V}(\mu),$$

where $\hat{V}(\mu)$ denotes the completion of $V(\mu)$ in the formal topology with respect to the weight decomposition of $V(\mu)$. (cf. [4]) The spin representations have level one prefect crystal (cf. [7]), then our model can be constructed by using the vertex operators with level one dominant integral weight λ and μ. Here we remark that the level one dominant integral weights are only $\Lambda_0,\ \Lambda_1,\ \Lambda_{n-1}$ and Λ_n in $D_n^{(1)}$ case. By using

conformal weight $\Delta_\lambda = (\lambda|\lambda+2\rho)/2(h^\vee+1)$ (for the level one dominant integral weights we have $\Delta_{\Lambda_0}=0$, $\Delta_{\Lambda_1}=1/2$, $\Delta_{\Lambda_{n-1}}=n/8$, $\Delta_{\Lambda_n}=n/8$), we multiply $z^{\Delta_\mu-\Delta_\lambda}$ to vertex operators as follows:

$$\Phi_\lambda^{\mu\, V^{(k)}}(z) = z^{\Delta_\mu-\Delta_\lambda}\tilde{\Phi}_\lambda^{\mu\, V^{(k)}}(z),$$
$$\Phi_\lambda^{V^{(k)}\,\mu}(z) = z^{\Delta_\mu-\Delta_\lambda}\tilde{\Phi}_\lambda^{V^{(k)}\,\mu}(z).$$

When we express the image of the highest weight vector $|\lambda\rangle \in V(\lambda)$ under $\tilde{\Phi}_\lambda^{\mu\, V^{(k)}}(z)$ as a linear combination of the weight vectors of $V(\mu)$, we call the coefficient of $|\mu\rangle$ the leading term of $\tilde{\Phi}_\lambda^{\mu\, V^{(k)}}(z)$ i.e.

$$\tilde{\Phi}_\lambda^{\mu\, V^{(k)}}(z)|\lambda\rangle = |\mu\rangle \otimes v + \cdots ,$$

where $v \in V^{(k)}$ is the leading term. Here we put

$$(V^{(k)})_\lambda^\mu = \{v \in V^{(k)} | \lambda \equiv \mu + \mathrm{wt}\, v \pmod{\delta},\ e_i{}^{\langle h_i,\mu\rangle+1}v = 0,\ \text{for all } i\}.$$

From the following proposition, we can know when non-trivial vertex operator exists.

Proposition 4.1. [4] *The mapping to send a vertex operator to its leading term gives an isomorphism of vector space:*

$$\{\tilde{\Phi}_\lambda^{\mu\, V^{(k)}}(z) \ :\ V(\lambda) \longrightarrow \hat{V}(\mu) \otimes V_z^{(k)}\} \simeq (V^{(k)})_\lambda^\mu.$$

Then for a given vector $v \in (V^{(k)})_\lambda^\mu$, a vertex operator which satisfies

$$\tilde{\Phi}_\lambda^{\mu\, V^{(k)}}(z)|\lambda\rangle = |\mu\rangle \otimes v + \cdots ,$$

uniquely exists. We will normalize vertex operators by specifying the leading term v.

In the case of $U_q(D_n^{(1)})$,

$$\dim (V^{(k)})_\lambda^\mu = 0 \text{ or } 1. \tag{28}$$

Therefore if there exists non-trivial vertex operators then it is unique up to multiple constant. Only in the following cases, non-trivial vertex operator $\tilde{\Phi}_\lambda^{\mu\, V^{(k)}}(z)$ exists. The existence and uniqueness conditions for type II vertex operators are exactly same (cf. [5]):

λ	μ	k
Λ_n	Λ_0	n
Λ_n	Λ_1	$n-1$
Λ_{n-1}	Λ_0	$n-1$
Λ_{n-1}	Λ_1	n
Λ_0	Λ_n	$n-i$
Λ_0	Λ_{n-1}	$n+i-1$
Λ_1	Λ_n	$n+i-1$
Λ_1	Λ_{n-1}	$n-i$

λ	μ	k
Λ_0	Λ_0	$1 \le k \le n-2:$ even
Λ_1	Λ_0	$1 \le k \le n-2:$ odd
Λ_1	Λ_1	$1 \le k \le n-2:$ even
Λ_0	Λ_1	$1 \le k \le n-2:$ odd
Λ_n	Λ_n	$1 \le k \le n-2:$ even
Λ_{n-1}	Λ_n	$1 \le k \le n-2:$ odd
Λ_{n-1}	Λ_{n-1}	$1 \le k \le n-2:$ even
Λ_n	Λ_{n-1}	$1 \le k \le n-2:$ odd

where $i=0$ if n is even and $i=1$ if n is odd.

After the normalization of three vertex operators $\tilde{\Phi}_{\Lambda_n}^{\Lambda_0\ V^{(n)}}(z)$, $\tilde{\Phi}_{\Lambda_0}^{\Lambda_0\ V^{(2k)}}(z)$ and $\tilde{\Phi}_{\Lambda_1}^{\Lambda_0\ V^{(2k-1)}}(z)$, the normalization of the others can be determined by using Dynkin diagram automorphisms in the same way as [5]. Here we can choose arbitrary normalization of the above three vertex operators, and so we fix the normalization as follows:

$$\begin{aligned}
\tilde{\Phi}_{\Lambda_n}^{\Lambda_0\ V^{(n)}}(z)|\Lambda_n\rangle &= |\Lambda_0\rangle \otimes v_1 + \cdots, \\
\tilde{\Phi}_{\Lambda_0}^{\Lambda_0\ V^{(k)}}(z)|\Lambda_0\rangle &= |\Lambda_0\rangle \otimes v_2 + \cdots \quad (k: \text{even}), \\
\tilde{\Phi}_{\Lambda_1}^{\Lambda_0\ V^{(k)}}(z)|\Lambda_1\rangle &= |\Lambda_0\rangle \otimes v_3 + \cdots \quad (k: \text{odd}).
\end{aligned}$$

The leading terms v_1, v_2 and v_3 are given by

$$\begin{aligned}
v_1 &= v_{(+,+,\cdots,+)}, \\
v_2 &= \sum_{m=0}^{\frac{k}{2}} c_m^{(n,k)} u_n^{(n-k+2m)}, \\
v_3 &= \sum_{m=0}^{\frac{k-1}{2}} c_m^{(n,k)} v_{++} \otimes u_{n-1}^{(n-k+2m)},
\end{aligned} \tag{29}$$

where

$$c_m^{(n,k)} = \prod_{j=0}^{m-1} \frac{q^{-2(j+1)}(1+q^{2m-2j})}{(1+q^{2n-2k+4j})(1+q^{2n-2k+4j+2})(1-q^{2n-2k+2j})},$$

and $u_n^{(k)}$ is inductively defined by

for $n \geq 2$,

$$\begin{aligned}
u_n^{(k)} &= -\frac{(-q)^{-n+k+2}}{(1+q^{2k})(1+q^{2k+2})}\{v_{+-} \otimes u_{n-1}^{(k+1)} - (-q)^{-k-1} v_{-+} \otimes \sigma_{n-1}\tilde{u}_{n-1}^{(k+1)}\} \\
&\quad +(-q)^{-n+k}\{v_{+-} \otimes u_{n-1}^{(k-1)} + (-q)^{k-1} v_{-+} \otimes \sigma_{n-1}\tilde{u}_{n-1}^{(k-1)}\} \qquad (1 \leq k \leq n), \\
\tilde{u}_n^{(k)} &= -\frac{(-q)^{-n+k+2}}{(1+q^{2k})(1+q^{2k+2})}\{v_{+-} \otimes \tilde{u}_{n-1}^{(k+1)} - (-q)^{-k-1} v_{-+} \otimes \sigma_{n-1} u_{n-1}^{(k+1)}\} \\
&\quad +(-q)^{-n+k}\{v_{+-} \otimes \tilde{u}_{n-1}^{(k-1)} + (-q)^{k-1} v_{-+} \otimes \sigma_{n-1} u_{n-1}^{(k-1)}\} \qquad (1 \leq k \leq n), \\
u_n^{(0)} &= -\frac{(-q)^{-n+2}}{(1+q^2)}\{v_{+-} \otimes u_{n-1}^{(1)} - (-q)^{-1} v_{-+} \otimes \sigma_{n-1}\tilde{u}_{n-1}^{(1)}\} \qquad (k=0), \\
\tilde{u}_n^{(0)} &= 0 \qquad (k=0), \\
u_n^{(k)} &= \tilde{u}_n^{(k)} = 0 \qquad (k > n \text{ or } k < 0),
\end{aligned}$$

for $n = 1$,

$$u_1^{(k)} = \tilde{u}_1^{(k)} = v_{+-} \; (k=1), \quad u_1^{(k)} = \tilde{u}_1^{(k)} = 0 \; (k \neq 1).$$

The normalization of type II vertex operators are given by

$$\begin{aligned}
\tilde{\Phi}_{\Lambda_n}^{V^{(n)}\ \Lambda_0}(z)|\Lambda_n\rangle &= v_1 \otimes |\Lambda_0\rangle + \cdots, \\
\tilde{\Phi}_{\Lambda_0}^{V^{(k)}\ \Lambda_0}(z)|\Lambda_0\rangle &= v_2 \otimes |\Lambda_0\rangle + \cdots \quad (k : \text{even}), \\
\tilde{\Phi}_{\Lambda_1}^{V^{(k)}\ \Lambda_0}(z)|\Lambda_1\rangle &= v_3 \otimes |\Lambda_0\rangle + \cdots \quad (k : \text{odd}),
\end{aligned}$$

where the leading term v_1, v_2 and v_3 are the same ones in (29). The normalization of the other cases is given by Dynkin diagram automorphisms.

4.2. **Vertex operators for dual modules.** By using the isomorphisms $C_\pm^{(k)}$ $(k = 1, 2, \cdots, n)$, we can construct vertex operators for dual modules. These isomorphisms are defined in (25) for $k = 1, 2, \cdots, n-2$ and $C_\pm^{(n-1)} = C_\pm^{(-)}$ and $C_\pm^{(n)} = C_\pm^{(+)}$ (see (14)). Here we denote by k' the number such that $V_{z\xi^{\mp 1}}^{(k')}$ is isomorphic to $(V_z^{(k)})^{*a^{\pm 1}}$ as $U_q(\mathfrak{g})$-module, i.e.

$$k' = \begin{cases} k & (1 \leq k \leq n-2), \\ n-1 & (n : \text{even},\ k = n-1), \\ n & (n : \text{even},\ k = n), \\ n & (n : \text{odd},\ k = n-1), \\ n-1 & (n : \text{odd},\ k = n). \end{cases}$$

Thanks to Proposition 4.4 and (28), the fact that $V_{z\xi^{\mp 1}}^{(k')}$ are isomorphic to $(V_z^{(k)})^{*a^{\pm 1}}$ leads that if there exists the vertex operator

$\Phi_\lambda^{\mu\ (V^{(k)})^{*a^{\pm 1}}}(z)$ then it is unique up to multiple scalar factor. Therefore we have

$$\tilde{\Phi}_\lambda^{\mu\ (V^{(k)})^{*a^{\pm 1}}}(z) = (const.)(\mathrm{id} \otimes C_\pm^{(k')})\tilde{\Phi}_\lambda^{\mu\ V^{(k')}}(z\xi^{\mp 1}),$$
$$\tilde{\Phi}_\lambda^{(V^{(k)})^{*a^{\pm 1}}\ \mu}(z) = (const.)(C_\pm^{(k')} \otimes \mathrm{id})\tilde{\Phi}_\lambda^{V^{(k')}\ \mu}(z\xi^{\mp 1}).$$

In order to construct the transfer matrix and the creation operator, it is enough to consider type I vertex operators $\Phi_\lambda^{\mu\ (V^{(k)})^{*a^{\pm 1}}}(z)$ for $k = n$ and type II vertex operators $\Phi_\lambda^{(V^{(k)})^{*a^{-1}}\ \mu}(z)$ for $k = 1, \dots, n$. (see Section 6)

Let $\{(v_{(\varepsilon_1,\dots,\varepsilon_n)})^*\}$ be dual basis of $\{v_{(\varepsilon_1,\dots,\varepsilon_n)}\}$. We give the normalization of the vertex operator $\tilde{\Phi}_{\Lambda_n}^{\Lambda_0\ (V^{(n)})^{*a^{\pm 1}}}(z)$ by

$$\tilde{\Phi}_{\Lambda_n}^{\Lambda_0\ (V^{(n)})^{*a^{\pm 1}}}(z)|\Lambda_n\rangle = |\Lambda_0\rangle \otimes (v_{(-,-,\cdots,-)})^* + \cdots,$$

and the other vertex operators are normalized by an appropriate Dynkin Diagram automorphism $\hat{\sigma}$ as follows:

$$\tilde{\Phi}_{\hat{\sigma}(\Lambda_n)}^{\hat{\sigma}(\Lambda_0)\ (\hat{\sigma}(V^{(n)}))^{*a^{\pm 1}}}(z)|\hat{\sigma}(\Lambda_n)\rangle = |\hat{\sigma}(\Lambda_0)\rangle \otimes (\hat{\sigma}v_{(-,-,\cdots,-)})^* + \cdots.$$

We normalize vertex operators of type II such that

$$\tilde{\Phi}_\lambda^{(V^{(k)})^{*a^{-1}}\ \mu}(z) = s(\mathrm{id} \otimes C_-^{(k')})\tilde{\Phi}_\lambda^{V^{(k')}\ \mu}(z\xi),$$

where s is a constant defined by

$$s = \begin{cases} (-1)^{[\frac{n}{2}]+k(n-k)} & (\lambda = \Lambda_0,\ \Lambda_1 \text{ and } k = n-1, n), \\ (-1)^{k(n-k)} & (\lambda = \Lambda_0,\ \Lambda_1 \text{ and } k \le n-2), \\ 1 & (\text{others}). \end{cases}$$

We remark that the constant s is defined as above so that the scalar factor $\tau^{(k)}(z)$ in the commutation relation (68) does not depend on the fundamental weights λ and μ.

4.3. **Two point functions.** We calculate the following vacuum expectation values of two vertex operators (two point functions).

$$\begin{array}{ll} \text{(type I–I)} & \langle \Phi_\mu^{\nu\ W_2}(z_2)\Phi_\lambda^{\mu\ V_1}(z_1)\rangle \\ \text{(type II–I)} & \langle \Phi_\mu^{W_2\ \nu}(z_2)\Phi_\lambda^{\mu\ V_1}(z_1)\rangle \\ \text{(type I–II)} & \langle \Phi_{\mu'}^{\nu\ W_2}(z_2)\Phi_\lambda^{V_1\ \mu'}(z_1)\rangle \end{array} \tag{30}$$

In the case of type I-I, consider the composition

$$V(\lambda) \xrightarrow{\Phi_\lambda^{\mu\ V}} \hat{V}(\mu) \otimes V_{z_1} \xrightarrow{\Phi_\mu^{\nu\ W} \otimes \mathrm{id}} \hat{V}(\nu) \otimes W_{z_2} \otimes V_{z_1} \xrightarrow{\mathrm{id} \otimes P} \hat{V}(\nu) \otimes V_{z_1} \otimes W_{z_2}.$$

Expressing the image of the highest weight vector of $V(\lambda)$ under the above composition as a linear combination of weight vectors with coefficients in $W_{z_1} \otimes V_{z_2}$, the coefficient of the highest weight vector of $V(\nu)$ is called vacuum expectation value. Here the subscript of the space $V^{(k)}$ and $V^{(n)}$ in two point function means that the space indexed by 1 (resp.2) always come in the first (resp. second) component of tensor product. Therefore in the case of type II-I and I-II we do not need the last transposition. (cf. [2])

Let $R_+^{V,W}(z)$ be the image of the modified R-matrix in (9).

Proposition 4.2. [2] *Let $\Psi(z_1, z_2)$ be a two point function of type I-I ,II-I or I-II. Then $\Psi(z_1, z_2)$ satisfies the following difference equation (the q-KZ equation):*

$$\begin{aligned} \Psi(pz_1, z_2) &= A(z_1/z_2)\Psi(z_1, z_2), \\ \Psi(pz_1, pz_2) &= (q^{-\phi} \otimes q^{-\phi})\Psi(z_1, z_2), \end{aligned} \tag{31}$$

where $p = q^{2(h^\vee+1)}$, $\phi = \bar{\lambda} + \bar{\nu} + 2\bar{\rho}$ and

$$\begin{aligned} A(z) &= R_+^{V,W}(pz)(q^{-\phi} \otimes 1) && \textit{for type I-I,} && (32) \\ &= (q^{-\bar{\nu}} \otimes 1)R_+^{V,W}(pq^{-1}z)(q^{-\phi+\bar{\nu}} \otimes 1) && \textit{for type II-I,} && (33) \\ &= (q^{-\phi+\bar{\nu}} \otimes 1)R_+^{V,W}(qz)(q^{-\bar{\nu}} \otimes 1) && \textit{for type I-II.} && (34) \end{aligned}$$

Then we can determine two point functions as solutions of the q-KZ equation.

For our aim to determine commutation relations of vertex operators, we need to calculate the following four types of two point functions:

$$\begin{array}{ll} \text{(i)} & \langle \Phi_{\mu}^{\nu\ (V^{(k)})_2}(z_2)\Phi_{\lambda}^{\mu\ (V^{(n)})_1}(z_1)\rangle \\ \text{(ii)} & \langle \Phi_{\mu'}^{\nu\ (V^{(n)})_2}(z_2)\Phi_{\lambda}^{\mu'\ (V^{(k)})_1}(z_1)\rangle \\ \text{(iii)} & \langle \Phi_{\mu'}^{\nu\ (V^{(n)})_2}(z_2)\Phi_{\lambda}^{(V^{(k)})_1\ \mu'}(z_1)\rangle \\ \text{(iv)} & \langle \Phi_{\mu}^{(V^{(k)})_1\ \nu}(z_1)\Phi_{\lambda}^{\mu\ (V^{(n)})_2}(z_2)\rangle \end{array} \tag{35}$$

where $k = 1, 2, \cdots, n$, and all combinations of weights $(\nu, \mu', \mu, \lambda)$ that non-trivial vertex operator exists are given by

For n: even

Table 1

ν	μ'	μ	λ	k
Λ_n	Λ_0	Λ_0	Λ_n	n
Λ_{n-1}	Λ_1	Λ_1	Λ_{n-1}	n
Λ_0	Λ_n	Λ_n	Λ_0	n
Λ_1	Λ_{n-1}	Λ_{n-1}	Λ_1	n

ν	μ'	μ	λ	k
Λ_n	Λ_0	Λ_1	Λ_{n-1}	$n-1$
Λ_{n-1}	Λ_1	Λ_0	Λ_n	$n-1$
Λ_0	Λ_n	Λ_{n-1}	Λ_1	$n-1$
Λ_1	Λ_{n-1}	Λ_n	Λ_0	$n-1$

ν	μ'	μ	λ	k
Λ_n	Λ_0	Λ_n	Λ_0	$1 \le k \le n-2$:even
Λ_{n-1}	Λ_1	Λ_{n-1}	Λ_1	$1 \le k \le n-2$:even
Λ_0	Λ_n	Λ_0	Λ_n	$1 \le k \le n-2$:even
Λ_1	Λ_{n-1}	Λ_1	Λ_{n-1}	$1 \le k \le n-2$:even

ν	μ'	μ	λ	k
Λ_n	Λ_0	Λ_{n-1}	Λ_1	$1 \le k \le n-2$:odd
Λ_{n-1}	Λ_1	Λ_n	Λ_0	$1 \le k \le n-2$:odd
Λ_0	Λ_n	Λ_1	Λ_{n-1}	$1 \le k \le n-2$:odd
Λ_1	Λ_{n-1}	Λ_0	Λ_n	$1 \le k \le n-2$:odd

For n: odd

Table 2

ν	μ'	μ	λ	k
Λ_n	Λ_1	Λ_1	Λ_{n-1}	n
Λ_{n-1}	Λ_0	Λ_0	Λ_n	n
Λ_0	Λ_n	Λ_n	Λ_1	n
Λ_1	Λ_{n-1}	Λ_{n-1}	Λ_0	n

ν	μ'	μ	λ	k
Λ_n	Λ_1	Λ_0	Λ_n	$n-1$
Λ_{n-1}	Λ_0	Λ_1	Λ_{n-1}	$n-1$
Λ_0	Λ_n	Λ_{n-1}	Λ_0	$n-1$
Λ_1	Λ_{n-1}	Λ_n	Λ_1	$n-1$

ν	μ'	μ	λ	k
Λ_n	Λ_1	Λ_n	Λ_1	$1 \le k \le n-2$:even
Λ_{n-1}	Λ_0	Λ_{n-1}	Λ_0	$1 \le k \le n-2$:even
Λ_0	Λ_n	Λ_0	Λ_n	$1 \le k \le n-2$:even
Λ_1	Λ_{n-1}	Λ_1	Λ_{n-1}	$1 \le k \le n-2$:even

ν	μ'	μ	λ	k
Λ_n	Λ_1	Λ_{n-1}	Λ_0	$1 \le k \le n-2$:odd
Λ_{n-1}	Λ_0	Λ_n	Λ_1	$1 \le k \le n-2$:odd
Λ_0	Λ_n	Λ_1	Λ_{n-1}	$1 \le k \le n-2$:odd
Λ_1	Λ_{n-1}	Λ_0	Λ_n	$1 \le k \le n-2$:odd

The following propositions in [2], [5] are useful to calculate two point functions.

Proposition 4.3. [2] *Let* $\Psi(z_1,z_2)$ *be a two point function of type* (i) *in* (35)*. For any* $i=0,1,\cdots,n$*,*

$$(\pi_{z_1}^{(n)}\otimes\pi_{z_2}^{(k)})\bigtriangleup'(e_i)^{\langle h_i,\nu\rangle+1}\Psi(z_1,z_2)=0,\quad \mathrm{wt}\,\Psi(z_1,z_2)=\bar{\lambda}-\bar{\nu},$$

hold.

Proposition 4.4. [2] *Let* $\Psi(z_1,z_2)$ *be a solution of the q-KZ equation* (31) *with* $A(z)$ *specified by* (32)*, then*

$$(q^{-\bar{\nu}}\otimes 1)\Psi(q^{-1}z_1,z_2),\quad (q^{-\phi+\bar{\nu}}\otimes 1)\Psi(p^{-1}qz_1,z_2),$$

satisfy the equation with $A(z)$ *specified by* (33) *and* (34) *respectively.*

Proposition 4.5. [5] *If a* $V\otimes W$*-valued function* $w(z)$ *satisfies*

$$(\pi_{z_1}^{V}\otimes\pi_{z_2}^{W})\Delta'(e_i)^{\langle h_i,\nu\rangle+1}w(z_1/z_2)=0\quad(i=0,1,\ldots,n),$$
$$\bar{R}^{V,W}(pz)(q^{-\phi}\otimes 1)w(z)=r(z)w(pz),$$

for some scalar function $r(z)$*, then* $\bar{w}(z)=P(q^{-\phi}\otimes 1)w(p^{-1}z^{-1})$ *satisfies*

$$(\pi_{z_1}^{W}\otimes\pi_{z_2}^{V})\Delta'(e_i)^{\langle h_i,\nu\rangle+1}\bar{w}(z_1/z_2)=0\quad(i=0,1,\ldots,n),$$
$$\bar{R}^{W,V}(pz)(q^{-\phi}\otimes 1)\bar{w}(z)=q^{\langle\phi,\mathrm{wt}\,\bar{w}\rangle}r(p^{-2}z^{-1})\bar{w}(pz).$$

Proposition 4.6. [5] *Let* λ*,* μ *be level 1 dominant integral weights and* v *be a weight vector in* $V^{(k)}$*. Define a non-negative integer* $m(\lambda,\mu;v)$ *by the minimal value of* m_0 *satisfying*

$$\lambda-\mu+\sum_{j=0}^{n}m_j\alpha_j\equiv\mathrm{wt}\,v\pmod{\mathbb{Z}\delta},$$
$$m_j\geq 0\ \ (j=0,1,\ldots,n).$$

If a two point function have a form

$$\langle\Phi_{\mu}^{\nu\,(V^{(\ell)})_2}(z_2)\Phi_{\lambda}^{\mu\,(V^{(k)})_1}(z_1)\rangle=z_1^{\Delta_\mu-\Delta_\lambda}z_2^{\Delta_\nu-\Delta_\mu}\sum_i a_i(z_1/z_2)v_i\otimes v_i',$$

then

$$\begin{aligned}\langle\Phi_{\hat{\sigma}(\mu)}^{\hat{\sigma}(\nu)\,\hat{\sigma}(V^{(\ell)})_2}(z_2)\Phi_{\hat{\sigma}(\lambda)}^{\hat{\sigma}(\mu)\,\hat{\sigma}(V^{(k)})_1}(z_1)\rangle &= z_1^{\Delta_{\widehat{\sigma(\mu)}}-\Delta_{\widehat{\sigma(\lambda)}}}z_2^{\Delta_{\widehat{\sigma(\nu)}}-\Delta_{\widehat{\sigma(\mu)}}}\\ &\quad\times\sum_i a_i(z_1/z_2)(z_1/z_2)^{m_i}\hat{\sigma}(v_i)\otimes\hat{\sigma}(v_i'),\end{aligned}$$

where $\hat{\sigma}$ *is a Dynkin diagram automorphism and*

$$m_i=m(\hat{\sigma}(\lambda),\hat{\sigma}(\mu);\hat{\sigma}(v_i))-m(\lambda,\mu;v_i).$$

Indeed, by using Proposition 4.4, type (iii) and (iv) in (35) are determined from type (i) and (ii). Furthermore we can calculate type (ii) from type (i) by virtue of Proposition 4.5. Hence we need to know explicit forms of type (i) for each case in Table 1 and Table 2. Thanks to Dynkin diagram symmetry of two point functions (Proposition 4.6), we obtain the two point functions in Table 1 and 2 from the ones listed below.

For n: even

	ν	μ'	μ	λ	k
case 1	Λ_0	Λ_n	Λ_n	Λ_0	n
case 2	Λ_0	Λ_n	Λ_{n-1}	Λ_1	$n-1$
case 3	Λ_0	Λ_n	Λ_0	Λ_n	$1 \le k \le n-2$:even
case 4	Λ_0	Λ_n	Λ_1	Λ_{n-1}	$1 \le k \le n-2$:odd

For n: odd

	ν	μ'	μ	λ	k
case 5	Λ_0	Λ_n	Λ_n	Λ_1	n
case 6	Λ_0	Λ_n	Λ_{n-1}	Λ_0	$n-1$
case 7	Λ_0	Λ_n	Λ_0	Λ_n	$1 \le k \le n-2$:even
case 8	Λ_0	Λ_n	Λ_1	Λ_{n-1}	$1 \le k \le n-2$:odd

Explicit forms of these two point functions are given in the next subsection.

4.4. **Explicit forms of two point functions.**

Case 1

(i) and (ii)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_0}^{\Lambda_n\ (V^{(n)})_1}(z_1)\rangle = z_1^{\frac{n}{8}} z_2^{-\frac{n}{8}} \psi^{(n,n)}(z_1/z_2) u_n^{(1)} \tag{36}$$

(iii)

$$\begin{aligned} &\langle \Phi_{\Lambda_n}^{(V^{(n)})_2\ \Lambda_0}(z_2)\Phi_{\Lambda_0}^{\Lambda_n\ (V^{(n)})_1}(z_1)\rangle \\ &\qquad = z_1^{\frac{n}{8}} z_2^{-\frac{n}{8}} \psi^{(n,n)}(p^{-1}qz_1/z_2)(-1)^{-\frac{n}{2}} q^{\frac{1}{2}n(n-1)} P u_n^{(1)} \end{aligned} \tag{37}$$

(iv)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_0}^{(V^{(n)})_1\ \Lambda_n}(z_1)\rangle = z_1^{-\frac{n}{8}} z_2^{\frac{n}{8}} \psi^{(n,n)}(q^{-1}z_2/z_1) P u_n^{(1)} \tag{38}$$

Case 2

(i)

$$\langle \Phi_{\Lambda_{n-1}}^{\Lambda_0\ (V^{(n-1)})_2}(z_2)\Phi_{\Lambda_1}^{\Lambda_{n-1}\ (V^{(n)})_1}(z_1)\rangle = z_1^{\frac{n}{8}-\frac{1}{2}} z_2^{-\frac{n}{8}} \psi^{(n,n-1)}(z_1/z_2)u_n^{(2)} \tag{39}$$

(ii)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_1}^{\Lambda_n\ (V^{(n-1)})_1}(z_1)\rangle = z_1^{\frac{n}{8}-\frac{1}{2}} z_2^{-\frac{n}{8}} \psi^{(n,n-1)}(z_1/z_2)(-1)^{-\frac{n}{2}-1} q^{\frac{1}{2}(2n-1)}(q^{-\phi}\otimes 1)Pu_n^{(2)} \tag{40}$$

(iii)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_1}^{(V^{(n-1)})_1\ \Lambda_n}(z_1)\rangle = z_1^{\frac{n}{8}-\frac{1}{2}} z_2^{-\frac{n}{8}} \psi^{(n,n-1)}(pq^{-1}z_1/z_2)(-1)^{-\frac{n}{2}+1} q^{-\frac{1}{2}(n-1)(n-2)} Pu_n^{(2)} \tag{41}$$

(iv)

$$\langle \Phi_{\Lambda_{n-1}}^{(V^{(n-1)})_2\ \Lambda_0}(z_2)\Phi_{\Lambda_1}^{\Lambda_{n-1}\ (V^{(n)})_1}(z_1)\rangle = z_1^{-\frac{n}{8}} z_2^{\frac{n}{8}-\frac{1}{2}} \psi^{(n,n-1)}(q^{-1}z_2/z_1)Pu_n^{(2)} \tag{42}$$

Case 3

(i)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_n\ (V^{(k)})_1}(z_1)\rangle = z_2^{-\frac{n}{8}} \psi^{(n,k)}(z_1/z_2)w^{(n,k)}(z_1/z_2) \tag{43}$$

(ii)

$$\langle \Phi_{\Lambda_0}^{\Lambda_0\ (V^{(k)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_1}(z_1)\rangle = z_1^{-\frac{n}{8}} \psi^{(n,k)}(p^{-1}z_2/z_1) q^{\frac{k}{2}(2n-1)}(z_1/z_2)^{\frac{k}{2}} P(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}z_2/z_1) \tag{44}$$

(iii)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_n}^{(V^{(k)})_1\ \Lambda_n}(z_1)\rangle = z_2^{-\frac{n}{8}} \psi^{(n,k)}(p^{-1}qz_1/z_2)(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2) \tag{45}$$

(iv)

$$\langle \Phi_{\Lambda_0}^{(V^{(k)})_1\ \Lambda_0}(z_2)\Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_1)\rangle = z_2^{-\frac{n}{8}} \psi^{(n,k)}(q^{-1}z_2/z_1) q^{k(n-1)}(z_1/z_2)^{-\frac{k}{2}}(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2) \tag{46}$$

Case 4

Two point functions for the case 4 are obtained from the following ones by using the Dynkin diagram automorphism $\hat{\sigma}_2$.

(i)

$$\langle \Phi_{\Lambda_{n-1}}^{\Lambda_0\ (V^{(n-1)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_{n-1}\ (V^{(k)})_1}(z_1)\rangle = z_2^{-\frac{n}{8}}\psi^{(n,k)}(z_1/z_2)w^{(n,k)}(z_1/z_2) \tag{47}$$

(ii)

$$\begin{aligned}\langle \Phi_{\Lambda_1}^{\Lambda_0\ (V^{(k)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_1\ (V^{(n-1)})_1}(z_1)\rangle \\ = z_1^{\frac{1}{2}-\frac{n}{8}}z_2^{-\frac{1}{2}}\psi^{(n,k)}(p^{-1}z_2/z_1) \\ \times(-1)^{n-1}q^{\frac{k}{2}(2n-1)}(z_1/z_2)^{\frac{k-1}{2}}P(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}z_2/z_1)\end{aligned} \tag{48}$$

(iii)

$$\begin{aligned}\langle \Phi_{\Lambda_{n-1}}^{\Lambda_0\ (V^{(n-1)})_2}(z_2)\Phi_{\Lambda_n}^{(V^{(k)})_1\ \Lambda_{n-1}}(z_1)\rangle \\ = z_2^{-\frac{n}{8}}\psi^{(n,k)}q^{\frac{1}{2}}(p^{-1}qz_1/z_2)(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2)\end{aligned} \tag{49}$$

(iv)

$$\begin{aligned}\langle \Phi_{\Lambda_1}^{(V^{(k)})_1\ \Lambda_0}(z_2)\Phi_{\Lambda_n}^{\Lambda_1\ (V^{(n-1)})_2}(z_1)\rangle \\ = z_1^{-\frac{1}{2}}z_2^{\frac{1}{2}-\frac{n}{8}}\psi^{(n,k)}(q^{-1}z_2/z_1) \\ \times(-1)^{n-1}q^{\frac{1}{2}}q^{k(n-1)}(z_1/z_2)^{-\frac{k-1}{2}}(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2)\end{aligned} \tag{50}$$

Case 5

(i) and (ii)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_1}^{\Lambda_n\ (V^{(n)})_1}(z_1)\rangle = z_1^{\frac{n}{8}-\frac{1}{2}}z_2^{-\frac{n}{8}}\psi^{(n,n)}(z_1/z_2)u_n^{(2)} \tag{51}$$

(iii)

$$\begin{aligned}\langle \Phi_{\Lambda_n}^{(V^{(n)})_2\ \Lambda_0}(z_2)\Phi_{\Lambda_0}^{\Lambda_n\ (V^{(n)})_1}(z_1)\rangle \\ = z_1^{\frac{n}{8}-\frac{1}{2}}z_2^{-\frac{n}{8}}\psi^{(n,n)}(-1)^{-\frac{n-1}{2}}q^{\frac{1}{2}(n-1)(n-2)}(p^{-1}qz_1/z_2)Pu_n^{(2)}\end{aligned} \tag{52}$$

(iv)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_0}^{(V^{(n)})_1\ \Lambda_n}(z_1)\rangle = z_1^{-\frac{n}{8}}z_2^{\frac{n}{8}-\frac{1}{2}}\psi^{(n,n)}(q^{-1}z_2/z_1)Pu_n^{(2)} \tag{53}$$

Case 6

(i)

$$\langle \Phi_{\Lambda_{n-1}}^{\Lambda_0\ (V^{(n-1)})_2}(z_2)\Phi_{\Lambda_0}^{\Lambda_{n-1}\ (V^{(n)})_1}(z_1)\rangle = z_1^{\frac{n}{8}} z_2^{-\frac{n}{8}}\psi^{(n,n-1)}(z_1/z_2)u_n^{(1)} \tag{54}$$

(ii)

$$\begin{aligned}&\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_0}^{\Lambda_n\ (V^{(n-1)})_1}(z_1)\rangle \\ &\qquad = z_1^{\frac{n}{8}} z_2^{-\frac{n}{8}}\psi^{(n,n-1)}(z_1/z_2)(-1)^{-\frac{n-1}{2}}q^{\frac{1}{2}(2n-1)}(q^{-\phi}\otimes 1)Pu_n^{(1)}\end{aligned} \tag{55}$$

(iii)

$$\begin{aligned}&\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_0}^{(V^{(n-1)})_1\ \Lambda_n}(z_1)\rangle \\ &\qquad = z_1^{\frac{n}{8}} z_2^{-\frac{n}{8}}\psi^{(n,n-1)}(pq^{-1}z_1/z_2)(-1)^{-\frac{n-1}{2}}q^{-\frac{n}{2}(n-1)}Pu_n^{(1)}\end{aligned} \tag{56}$$

(iv)

$$\langle \Phi_{\Lambda_{n-1}}^{(V^{(n-1)})_2\ \Lambda_0}(z_2)\Phi_{\Lambda_0}^{\Lambda_{n-1}\ (V^{(n)})_1}(z_1)\rangle = z_1^{-\frac{n}{8}} z_2^{\frac{n}{8}}\psi^{(n,n-1)}(q^{-1}z_2/z_1)Pu_n^{(1)} \tag{57}$$

Case 7

(i)

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_n\ (V^{(k)})_1}(z_1)\rangle = z_2^{-\frac{n}{8}}\psi^{(n,k)}(z_1/z_2)w^{(n,k)}(z_1/z_2) \tag{58}$$

(ii)

$$\begin{aligned}&\langle \Phi_{\Lambda_0}^{\Lambda_0\ (V^{(k)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_1}(z_1)\rangle \\ &\quad = z_1^{-\frac{n}{8}}\psi^{(n,k)}(p^{-1}z_2/z_1)q^{\frac{k}{2}(2n-1)}(z_1/z_2)^{\frac{k}{2}}P(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}z_2/z_1)\end{aligned} \tag{59}$$

(iii)

$$\begin{aligned}&\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_n}^{(V^{(k)})_1\ \Lambda_n}(z_1)\rangle \\ &\qquad = z_2^{-\frac{n}{8}}\psi^{(n,k)}(p^{-1}qz_1/z_2)(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2)\end{aligned} \tag{60}$$

(iv)

$$\begin{aligned}&\langle \Phi_{\Lambda_0}^{(V^{(k)})_1\ \Lambda_0}(z_2)\Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_1)\rangle \\ &\quad = z_2^{-\frac{n}{8}}\psi^{(n,k)}(q^{-1}z_2/z_1)q^{k(n-1)}(z_1/z_2)^{-\frac{k}{2}}(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2)\end{aligned} \tag{61}$$

Case 8

Two point functions for the case 8 are obtained from the following ones by using the Dynkin diagram automorphism $\hat{\sigma}_2$.

(i)

$$\langle \Phi_{\Lambda_{n-1}}^{\Lambda_0\ (V^{(n-1)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_{n-1}\ (V^{(k)})_1}(z_1)\rangle = z_2^{-\frac{n}{8}}\psi^{(n,k)}(z_1/z_2)w^{(n,k)}(z_1/z_2) \tag{62}$$

(ii)

$$\begin{aligned}&\langle \Phi_{\Lambda_1}^{\Lambda_0\ (V^{(k)})_2}(z_2)\Phi_{\Lambda_n}^{\Lambda_1\ (V^{(n-1)})_1}(z_1)\rangle \\ &\quad = z_1^{\frac{1}{2}-\frac{n}{8}}z_2^{-\frac{1}{2}}\psi^{(n,k)}(p^{-1}z_2/z_1) \\ &\quad\quad \times (-1)^{n-1}q^{\frac{k}{2}(2n-1)}(z_1/z_2)^{\frac{k-1}{2}}P(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}z_2/z_1)\end{aligned} \tag{63}$$

(iii)

$$\begin{aligned}&\langle \Phi_{\Lambda_{n-1}}^{\Lambda_0\ (V^{(n-1)})_2}(z_2)\Phi_{\Lambda_n}^{(V^{(k)})_1\ \Lambda_{n-1}}(z_1)\rangle \\ &\quad = z_2^{-\frac{n}{8}}\psi^{(n,k)}q^{\frac{1}{2}}(p^{-1}qz_1/z_2)(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2)\end{aligned} \tag{64}$$

(iv)

$$\begin{aligned}&\langle \Phi_{\Lambda_1}^{(V^{(k)})_1\ \Lambda_0}(z_2)\Phi_{\Lambda_n}^{\Lambda_1\ (V^{(n-1)})_2}(z_1)\rangle \\ &\quad = z_1^{-\frac{1}{2}}z_2^{\frac{1}{2}-\frac{n}{8}}\psi^{(n,k)}(q^{-1}z_2/z_1) \\ &\quad\quad \times (-1)^{n-1}q^{\frac{1}{2}}q^{k(n-1)}(z_1/z_2)^{-\frac{k-1}{2}}(q^{-\phi}\otimes 1)w^{(n,k)}(p^{-1}qz_1/z_2)\end{aligned} \tag{65}$$

where $\psi^{(n,k)}(z)$ is given by

$$\psi^{(n,k)}(z) = \begin{cases} \displaystyle\prod_{j=1}^{k-1}\frac{(-q\xi^{5/2}q^{2j-k+1}(-1)^{n-k}z_1/z_2;\xi^2)_\infty}{(-q\xi^{3/2}q^{2j-k+1}(-1)^{n-k}z_1/z_2;\xi^2)_\infty} & (1\le k\le n-2), \\ \displaystyle\prod_{i=1}^{[\frac{n-1}{2}]}\frac{(q^{4i-2}pz_1/z_2;\xi^2)_\infty}{(q^{-4i}pz_1/z_2;\xi^2)_\infty} & (k=n-1), \\ \displaystyle\prod_{i=1}^{[\frac{n}{2}]}\frac{(q^{4i-4}pz_1/z_2;\xi^2)_\infty}{(q^{-4i+2}pz_1/z_2;\xi^2)_\infty} & (k=n). \end{cases}$$

Furthermore, $u_n^{(1)}$, $u_n^{(2)}$ and $w^{(n,k)}(z)$ are given as follows:

$$\begin{aligned} u_n^{(1)} &= (-q)^{\frac{1}{2}n(n-1)}v_n, \\ u_n^{(2)} &= (-q)^{\frac{1}{2}(n-1)(n-2)}v_{++}\otimes v_{n-1}, \end{aligned}$$

where v_n is inductively defined by

$$v_n = v_{+-} \otimes v_{n-1} + (-q)^{-n+1} v_{-+} \otimes \sigma_{n-1} v_{n-1}, \quad v_1 = v_{+-}.$$

$$w^{(n,k)}(z) = \sum_{m=0}^{[\frac{k}{2}]} c_m^{(n,k)}(z) w_n^{(n-k-2m)},$$

$$\begin{aligned} c_m^{(n,k)}(z) = \prod_{j=0}^{m-1} & \frac{q^{-2(j+1)}}{(1+q^{2n-2k+4j})(1+q^{2n-2k+4j+2})(1-q^{2n-2k+2j})} \\ & \times \sum_{i=0}^{m} \{ \prod_{j=0}^{m-1-i} (1+q^{2m-2j}) \prod_{j=0}^{i-1} (1+q^{2n-2k+2m+2j}) \begin{bmatrix} m \\ i \end{bmatrix}_q ((-1)^{n-k} q^{n+k-m} z)^i \}, \end{aligned}$$

and $w_n^{(k)}$ is inductively defined by for $n \geq 2$

$$\begin{aligned} w_n^{(k)} = & \frac{-(-q)^{-n+k+2}}{(1+q^{2k})(1+q^{2k+2})} \{ v_{+-+} \otimes w_{n-1}^{(k+1)} - (-q)^{-k-1} v_{-++} \otimes (\sigma_{n-1})_{12} \tilde{w}_{n-1}^{(k+1)} \} \\ & + v_{++-} \otimes w_{n-1}^{(k)} \qquad (1 \leq k \leq n), \\ & + (-q)^{-n+k} \{ v_{+-+} \otimes w_{n-1}^{(k-1)} + (-q)^{k-1} v_{-++} \otimes (\sigma_{n-1})_{12} \tilde{w}_{n-1}^{(k-1)} \} \end{aligned}$$

$$\begin{aligned} \tilde{w}_n^{(k)} = & \frac{-(-q)^{-n+k+2}}{(1+q^{2k})(1+q^{2k+2})} \{ v_{+-+} \otimes \tilde{w}_{n-1}^{(k+1)} - (-q)^{-k-1} v_{-++} \otimes (\sigma_{n-1})_{12} w_{n-1}^{(k+1)} \} \\ & + v_{++-} \otimes \tilde{w}_{n-1}^{(k)} \qquad (1 \leq k \leq n), \\ & + (-q)^{-n+k} \{ v_{+-+} \otimes \tilde{w}_{n-1}^{(k-1)} + (-q)^{k-1} v_{-++} \otimes (\sigma_{n-1})_{12} w_{n-1}^{(k-1)} \} \end{aligned}$$

$$\begin{aligned} w_n^{(0)} = & \frac{-(-q)^{-n+2}}{(1+q^2)} \{ v_{+-+} \otimes w_{n-1}^{(1)} - (-q)^{-1} v_{-++} \otimes (\sigma_{n-1})_{12} \tilde{w}_{n-1}^{(1)} \} \\ & + v_{++-} \otimes w_{n-1}^{(0)} \qquad (k=0), \\ & \tilde{w}_n^{(0)} = 0 \qquad (k=0), \\ & w_n^{(k)} = \tilde{w}_n^{(k)} = 0 \qquad (k > n \text{ or } k < 0), \end{aligned}$$

for $n = 1$

$$\begin{aligned} w_1^{(k)} = \tilde{w}_1^{(k)} = v_{+-+} & \quad (k=1), \\ w_1^{(k)} = v_{++-}, \ \ \tilde{w}_1^{(k)} = 0 & \quad (k=0), \\ w_1^{(k)} = \tilde{w}_1^{(k)} = 0 & \quad (k \neq 0, \ 1). \end{aligned}$$

Here recursive formula of $w_n^{(k)}$ and $\tilde{w}_n^{(k)}$ are understood as similar to Section 2.5 and $w_n^{(k)}$ denotes an equivalent class represented by itself

in the space

$$\{V^{(n)} \otimes V^{(n')} \otimes V^{(m)}\}/\{\ker T^{(k)} \otimes V^{(m)}\} = V^{(k)} \otimes V^{(m)},$$

where $n' = \varphi^{(n)}(n-k)$ and $m = \varphi^{(n)}(k)$.

5. Commutation relations

We describe commutation relations of vertex operators and make a remark on difference equations for scalar functions which appeared in the commutation relations.

From the explicit forms of two point functions, we can obtain the following theorem.

Theorem 5.1. *For any possible combination of weights* $(\nu, \mu', \mu, \lambda)$ *(see Table 1 and 2 in Section 4.3),*

$$\Phi_{\mu}^{\nu\ V_2^{(n)}}(z_2)\Phi_{\lambda}^{\mu\ V_1^{(n)}}(z_1) = r_n(z_1/z_2)\bar{R}_n^{(+,+)}(z_1/z_2)\Phi_{\mu}^{\nu\ V_1^{(n)}}(z_1)\Phi_{\lambda}^{\mu\ V_2^{(n)}}(z_2), \tag{66}$$

$$\Phi_{\mu'}^{\nu\ V_2^{(n)}}(z_2)\Phi_{\lambda}^{V_1^{(k)}\ \mu'}(z_1) = \tau^{(k)}(z_1/z_2)\Phi_{\mu}^{V_1^{(k)}\ \nu}(z_1)\Phi_{\lambda}^{\mu\ V_2^{(n)}}(z_2), \tag{67}$$

$$\Phi_{\mu'}^{\nu\ V_2^{(n)}}(z_2)\Phi_{\lambda}^{V_1^{(k)*a^{-1}}\ \mu'}(z_1) = \tau^{(k)}(z_1/z_2)^{-1}\Phi_{\mu}^{V_1^{(k)*a^{-1}}\ \nu}(z_1)\Phi_{\lambda}^{\mu\ V_2^{(n)}}(z_2), \tag{68}$$

where $r_n(z)$ *and* $\tau^{(k)}(z)$ *are given by*

$$r_n(z) = z^{-\frac{n}{4}} \prod_{i=1}^{[\frac{n}{2}]} \frac{(q^{4i-2}z;\xi^2)_\infty (q^{4n-4i}z^{-1};\xi^2)_\infty}{(q^{4i-2}z^{-1};\xi^2)_\infty (q^{4n-4i}z;\xi^2)_\infty},$$

$$\tau^{(k)}(z) = \begin{cases} z^{-\frac{k}{2}} \displaystyle\prod_{j=1}^{k} \frac{\Theta_{\xi^2}(-(-q)^{k+n-2j}z)}{\Theta_{\xi^2}(-(-q)^{k+n-2j}z^{-1})} & (1 \le k \le n-2), \\ z^{-\frac{n-2}{4}} \displaystyle\prod_{j=1}^{[\frac{n-1}{2}]} \frac{\Theta_{\xi^2}(-(-q)^{4j-1}z)}{\Theta_{\xi^2}(-(-q)^{4j-1}z^{-1})} & (k = n-1), \\ z^{-\frac{n}{4}} \displaystyle\prod_{j=1}^{[\frac{n}{2}]} \frac{\Theta_{\xi^2}(-(-q)^{4j-3}z)}{\Theta_{\xi^2}(-(-q)^{4j-3}z^{-1})} & (k = n). \end{cases}$$

To prove this theorem, it is enough to show that the vacuum expectation values of the both sides in commutation relations are exactly same. Indeed, by irreducibility of $V(\lambda)$ and $V(\nu)$, if the vacuum expectation values of both sides coincide, then the equality as operators on $V(\lambda)$ can be proved.

This theorem gives us the explicit forms of the excitation spectra mentioned in Introduction.

We make a remark on a relation between difference equations for the scalar function $\tau^{(k)}(z)$ and fusion procedure of finite dimensional $U_q(D_n^{(1)})$-modules (cf. [9]).

Remark 5.1. *The scalar factor $\tau^{(k)}(z)$ satisfies the following difference equations:*
for $n+k$:even

$$\tau^{(k)}(z) = (-1)^{-\frac{n}{2}}\tau^{(n)}(\zeta^{-1}z)\tau^{(n)}(\zeta z),$$
$$\tau^{(k)}(z) = (-1)^{-\frac{n}{2}+1}\tau^{(n-1)}(\zeta^{-1}z)\tau^{(n-1)}(\zeta z),$$

for $n+k$:odd

$$\tau^{(k)}(z) = (-1)^{-\frac{1}{2}(n-k-1)}\tau^{(n)}(\zeta^{-1}z)\tau^{(n-1)}(\zeta z),$$
$$\tau^{(k)}(z) = (-1)^{-\frac{1}{2}(n-k-1)}\tau^{(n-1)}(\zeta^{-1}z)\tau^{(n)}(\zeta z),$$

where we put $\zeta = (-q)^{n-k-1}$. We can see correspondence between these equations and the following fusion procedure

$$V_z^{(k)} \hookrightarrow V_{\zeta^{-1}z}^{(n)} \otimes V_{\zeta z}^{(n)} \simeq V_{\zeta^{-1}z}^{(n-1)} \otimes V_{\zeta z}^{(n-1)} \qquad (n+k : \textit{even}),$$
$$V_z^{(k)} \hookrightarrow V_{\zeta^{-1}z}^{(n)} \otimes V_{\zeta z}^{(n-1)} \simeq V_{\zeta^{-1}z}^{(n-1)} \otimes V_{\zeta z}^{(n)} \qquad (n+k : \textit{odd}).$$

6. Formulation of the model

We will construct the transfer matrix, the creation and annihilation operators, in the same way as in [1].

In quantum symmetry approach, we define the space of states by

$$\mathcal{F} = \bigoplus_{\lambda,\mu\in\{\Lambda_0,\Lambda_1,\Lambda_{n-1},\Lambda_n\}} \mathcal{F}_{\lambda,\mu},$$

where

$$\mathcal{F}_{\lambda,\mu} := V(\lambda)\otimes V(\mu)^{*a} \equiv \mathrm{Hom}_{\mathbb{Q}(q)}(V(\mu),V(\lambda)).$$

Following [1], we complete the space $\mathcal{F}_{\lambda,\mu}$ in the topology of formal power series in q. From now on we denote the completed space by the same simbol $\mathcal{F}_{\lambda,\mu}$.

The left $U_q(\mathfrak{g})$-module structure of $\mathcal{F}_{\lambda,\mu}$ can be written as

$$xf = \sum x_{(1)}\circ f\circ a(x_{(2)}) \quad (f\in\mathcal{F}_{\lambda,\mu},\ x\in U_q(\mathfrak{g})),$$

where $\triangle(x) = \sum x_{(1)}\otimes x_{(2)}$. At the same time, we can define right module structure on $\mathcal{F}_{\lambda,\mu}$ by

$$fx = \sum a^{-1}(x_{(2)})\circ f\circ x_{(1)},$$

and then we denote this right module by $\mathcal{F}^r_{\lambda,\mu}$. We have a natural pairing (cf. [2])

$$\langle f|g\rangle := \frac{\mathrm{tr}_{V(\lambda)}(q^{-2\rho}fg)}{\mathrm{tr}_{V(\lambda)}(q^{-2\rho})}, \quad f\in\mathcal{F}^r_{\lambda,\mu},\ g\in\mathcal{F}_{\mu,\lambda}. \tag{69}$$

which satisfies

$$\langle fx|g\rangle = \langle f|xg\rangle.$$

In order to define "local operators" on our space of states, we consider vertex operator

$$\tilde{\Phi}^{\mu}_{\lambda\ V^{(n)}}(z)\ :\ V(\lambda)\longrightarrow \hat{V}(\mu)\otimes V^{(k)}_z.$$

We fix the normalization of $\tilde{\Phi}^{\mu}_{\lambda\ V^{(n)}}(z)$ by

$$\tilde{\Phi}^{\Lambda_0}_{\Lambda_n\ V^{(n)}}(z)(|\Lambda_n\rangle\otimes v_{(+,+,\cdots,+)}) = |\Lambda_0\rangle+\cdots,$$

and normalize the other vertex operators by using Dynkin diagram automorphisms.

Then we can show the following formula (cf. [2])

$$\begin{aligned}\tilde{\Phi}^{\mu}_{\lambda\ V^{(n)}}(z)\tilde{\Phi}^{\mu\ V^{(n)}}_{\lambda}(z) &= g\times\mathrm{id}_{V(\lambda)},\\ \tilde{\Phi}^{\lambda\ V^{(n)}}_{\mu}(z)\tilde{\Phi}^{\lambda}_{\mu\ V^{(n)}}(z) &= g\times\mathrm{id}_{V(\lambda)\otimes V^{(n)}}.\end{aligned}$$

where

$$g=\begin{cases}\displaystyle\prod_{i=1}^{\frac{n}{2}}\frac{(q^{4i-2}\xi;\xi^2)_\infty}{(q^{-4i+4}\xi;\xi^2)_\infty} & (n:\text{even}),\\[2ex] \displaystyle\prod_{i=1}^{\frac{n-1}{2}}\frac{(q^{4i}\xi;\xi^2)_\infty}{(q^{-4i+2}\xi;\xi^2)_\infty} & (n:\text{odd}).\end{cases}$$

These formulas can be proved by the explicit forms of two point functions in Section 4.4 in the same way as [2].

By the formulas, we see that

$$\tilde{\Phi}^{\mu\ V^{(n)}}_{\lambda}(z)\ :\ V(\lambda)\longrightarrow\hat{V}(\mu)\otimes V^{(n)}_z$$

is invertible (cf. [1]).

We recall that, for any integral weight λ of level one, there is a unique level one weight μ such that non-trivial vertex operator $\tilde{\Phi}^{\mu\ V^{(k)}}_{\lambda}(z)$ exists (see Section 4.1). We denote such μ by $\lambda^{(k)}$. For vertex operator $\tilde{\Phi}^{\mu\ V^{(k)*a^{\pm1}}}_{\lambda}(z)$, we define $\lambda^{(k)*}$ similarly.

The row transfer matrix

$$T^{\lambda^{(n)},\mu^{(n)}}_{\lambda,\mu}\ :\ \mathcal{F}_{\lambda,\mu}\longrightarrow\mathcal{F}_{\lambda^{(n)},\mu^{(n)}},$$

is defined by the composition of the following operators

$$V(\lambda) \otimes V(\mu)^{*a} \longrightarrow V(\lambda^{(n)}) \otimes V_z^{(n)} \otimes V(\mu)^{*a} \longrightarrow V(\lambda^{(n)}) \otimes V(\mu^{(n)})^{*a}, \tag{70}$$

where the first map is $\tilde{\Phi}_{\lambda}^{\lambda^{(n)}\ V^{(n)}}(z) \otimes \mathrm{id}$ and the second one is $\mathrm{id} \otimes (\tilde{\Phi}_{\mu^{(n)}}^{\mu\ V^{(n)*a^{-1}}}(z))^t$.

Since $\mathrm{id}_{V(\lambda)} \in \mathcal{F}_{\lambda,\lambda}$, we define $|\mathrm{vac}\rangle_\lambda := \mathrm{id}_{V(\lambda)}$. In a similar way to [2], we can show

$$T_{\lambda,\lambda}^{\lambda^{(n)},\lambda^{(n)}} |\mathrm{vac}\rangle_\lambda = g|\mathrm{vac}\rangle_{\lambda^{(n)}},$$

from the next formula

$$\bar{p}(\langle \tilde{\Phi}_{\mu}^{\lambda V_1^{(n)}}(z) \tilde{\Phi}_{\lambda}^{\mu V_2^{(n)*a^{-1}}}(z) \rangle) \quad = \quad g,$$

where

$$\begin{aligned} \bar{p} \ : \ V^{(n)} \otimes V^{(n)*a^{-1}} &\quad \longrightarrow \quad \mathbb{Q}(q), \\ v_1 \otimes v_2^* &\quad \mapsto \quad \langle v_1, v_2^* \rangle. \end{aligned}$$

The creation and annihilation operators are constructed by using type II vertex operators. We express vertex operators as follows:

$$\begin{aligned} \Phi_{\lambda}^{V^{(k)}\ \lambda^{(k)}}(z) &\quad = \quad \sum_I v_I \otimes \Phi_{\lambda,I}^{(k)}(z), \\ \Phi_{\lambda}^{V^{(k)*a^{-1}}\ \lambda^{(k)*}}(z) &\quad = \quad \sum_I v_I^* \otimes \Phi_{\lambda,I}^{(k)*}(z), \end{aligned}$$

where $\{v_I\}$ and $\{v_I^*\}$ are dual basis of $V^{(k)}$ and $V^{(k)*a^{-1}}$ and $\Phi_{\lambda,I}^{(k)}(z),\ \Phi_{\lambda,I}^{(k)*}(z) \in \mathcal{F}_{\lambda,\lambda^{(k)*}}$. The creation operator

$$\phi_{\lambda,I}^{(k)*}(z) \ : \ \mathcal{F}_{\lambda,\mu} \longrightarrow \mathcal{F}_{\lambda^{(k)*},\mu},$$

is given by

$$f \ \mapsto \Phi_{\lambda,I}^{(k)*}(z) \circ f.$$

The annihilation operator

$$\phi_{\lambda,I}^{(k)}(z) \ : \ \mathcal{F}_{\lambda,\mu} \longrightarrow \mathcal{F}_{\lambda^{(k)*},\mu},$$

is defined by the adjoint of

$$\mathcal{F}_{\mu,\lambda^{(k)}}^{r} \longrightarrow \mathcal{F}_{\mu,\lambda}^{r}, \quad f \ \mapsto f \circ \Phi_{\lambda,I}^{(k)}(z),$$

with respect to the pairing (69).

Commutation relations of vertex operators (67) and (68) lead the following relations between the transfer matrix and the creation and

annihilation operators:

$$\begin{aligned}\phi^{(k)*}_{\lambda^{(n)*},I}(z)T^{\lambda^{(n)},\mu^{(n)}}_{\lambda,\mu} &= \tau^{(k)}(z_1/z_2)T^{(\lambda^{(k)*})^{(n)},\mu^{(n)}}_{\lambda^{(k)*},\mu}\phi^{(k)*}_{\lambda,I}(z),\\ \phi^{(k)*}_{\lambda^{(n)},I}(z)T^{\lambda^{(n)},\mu^{(n)}}_{\lambda,\mu} &= \tau^{(k)}(z_1/z_2)^{-1}T^{(\lambda^{(k)*})^{(n)},\mu^{(n)}}_{\lambda^{(k)*},\mu}\phi^{(k)}_{\lambda,I}(z),\end{aligned}$$

where $\tau^{(k)}(z_1/z_2)$ is given in the previous section.

7. Appendix A

We calculate the scalar function $\alpha_n^{(\varepsilon_1,\varepsilon_2)}(z)$ in (21) and $\beta_n^{(\varepsilon_1,\varepsilon_2)}(z)$ in (22). We only consider $\alpha_n^{(+,+)}(z)$ and $\beta_n^{(+,+)}(z)$ for any even integer n. (The other cases can be calculated similarly.)

7.1. Scalar function $\alpha_n^{(+,+)}(z)$.

The second inversion relation can be written as follows:

$$\alpha_n^{(+,+)}(z)^{-1}(\bar{R}_n^{(+,+)}(z)^{-1})^{t_1} = (q^{2\rho}\otimes 1)(\bar{R}_n^{(+,+)}(\xi^{-2}z)^{t_1})^{-1}(q^{-2\rho}\otimes 1). \tag{71}$$

Let v_n be a vector $v_{(+,\cdots,+)}\otimes v_{(-,\cdots,-)} \in V^{(+)}\otimes V^{(+)}$. Applying both sides in (71) to v_n, we can obtain the explicit form of $\alpha_n^{(+,+)}(z)$. We will show that v_n is an eigenvector for operators $(q^{2\rho}\otimes 1)(\bar{R}_n^{(+,+)}(\xi^{-2}z)^{t_1})^{-1}(q^{-2\rho}\otimes 1)$ (resp. $(\bar{R}_n^{(+,+)}(z)^{-1})^{t_1}$) and its eigenvalue is given by $\prod_{i=1}^{\frac{n}{2}} a(q^{4i-4}\xi^{-2}z)$ (resp. $\prod_{i=1}^{\frac{n}{2}} a(q^{4-4i}z^{-1})$).

By using the isomorphism (16), we see $v_n = v_{+-}\otimes v_{n-1}$ where $v_{n-1}\in V_{n-1}^{(+)}\otimes V_{n-1}^{(-)}$. Taking transpose in the first component of the tensor product in the recursive formulae (18), we can obtain recursive relations for $\bar{R}_n^{(+,+)}(z)^{t_1}$. In particular we have

$$\begin{aligned}\bar{R}_n^{(+,+)}(z)^{t_1}v_n &= a(z)v_{+-}\otimes \bar{R}_{n-1}^{(+,-)}(q^2z)^{t_1}v_{n-1}, && (72)\\ \bar{R}_n^{(+,-)}(z)^{t_1}v_n &= v_{+-}\otimes \bar{R}_{n-1}^{(+,+)}(q^2z)^{t_1}v_{n-1}, && (73)\end{aligned}$$

where $a(z) = q(1-z)/(1-q^2z)$. Combining (72) with (73), we find

$$\bar{R}_n^{(+,+)}(z)^{t_1}v_n = a(z)v_{+-}\otimes v_{+-}\otimes \bar{R}_{n-2}^{(+,+)}(q^4z)^{t_1}v_{n-2},$$

then

$$\begin{aligned}\bar{R}_n^{(+,+)}(z)^{t_1}(v_n) &= \prod_{i=1}^{\frac{n}{2}} a(q^{4i-4}z)v_{+-}\otimes v_{+-}\otimes\cdots\otimes v_{+-}\\ &= \prod_{i=1}^{\frac{n}{2}} a(q^{4i-4}z)v_n.\end{aligned}$$

Since $(q^{-2\bar{\rho}} \otimes 1)v_n = q^{-n(n-1)/2}v_n$, we obtain

$$(q^{2\rho} \otimes 1)(\bar{R}_n^{(+,+)}(\xi^{-2}z)^{t_1})^{-1}(q^{-2\rho} \otimes 1)v_n = \prod_{i=1}^{\frac{n}{2}} a(q^{4i-4}\xi^{-2}z)^{-1}v_n.$$

We can similarly show

$$(\bar{R}_n^{(+,+)}(z)^{-1})^{t_1}v_n = \prod_{i=1}^{\frac{n}{2}} a(q^{4i-4}z^{-1})v_n,$$

by using the first inversion relation

$$\bar{R}_n^{(+,+)}(z)^{-1} = P\bar{R}_n^{(+,+)}(z^{-1})P \quad (P : \text{transposition}).$$

Applying both sides in (71) to v_n, we obtain

$$\begin{aligned}\alpha_n^{(+,+)}(z) &= \prod_{i=1}^{\frac{n}{2}} a(q^{4i-4}z^{-1})a(\xi^{-2}q^{4i-4}z)\\ &= \prod_{i=1}^{\frac{n}{2}} \frac{(1-q^{-4i+4}z)(1-q^{-4n+4i}z)}{(1-q^{-4i+2}z)(1-q^{-4n+4i+2}z)}.\end{aligned}$$

7.2. **Scalar factor $\beta_n^{(+,+)}(z)$.** We can determine the explicit form of the scalar factor $\beta_n^{(+,+)}(z)$ in (22) by using $\alpha_n^{(+,+)}(z)$.

We recall that

$$(\pi_{z_1}^{(+)} \otimes \pi_{z_2}^{(+)})(\mathcal{R}'(z)) = \beta_n^{(+,+)}(z)\bar{R}_n^{(+,+)}(z). \tag{74}$$

and the factor $\beta_n^{(+,+)}(z)$ is a solution of the following difference equation (see Section 2.3):

$$\alpha_n^{(+,+)}(z)\beta_n^{(+,+)}(z)^{-1} = \beta_n^{(+,+)}(\xi^{-2}z)^{-1}.$$

Then $\beta_n^{(+,+)}(z)$ can be written as

$$\beta_n^{(+,+)}(z) = c\prod_{i=1}^{\infty} \alpha_n^{(+,+)}(z\xi^{2i})^{-1} = c\prod_{i=1}^{\frac{n}{2}} \frac{(q^{4n-4i-2}z;\xi^2)_\infty(q^{4i-2}z;\xi^2)_\infty}{(q^{4n-4i}z;\xi^2)_\infty(q^{4i-4}z;\xi^2)_\infty}.$$

for some constant c. Applying the both sides in (74) to $v_{(+,\cdots,+)} \otimes v_{(+,\cdots,+)}$ we can determine the constant c. Indeed, from the explicit form of the modified universal R-matrix in (9), we find

$$\begin{aligned}&(\pi_{z_1}^{(+)} \otimes \pi_{z_2}^{(+)})(\mathcal{R}'(z))(v_{(+,\cdots,+)} \otimes v_{(+,\cdots,+)})\\ &\quad = q^{-\sum_{i=1}^n \pi^{(+)}(h_i)\otimes\pi^{(+)}(\bar{\Lambda}_i)}(v_{(+,\cdots,+)} \otimes v_{(+,\cdots,+)}) + (\text{higher degree of } z),\end{aligned}$$

and by direct calculation we have

$$q^{-\sum_{i=1}^n \pi^{(+)}(h_i)\otimes\pi^{(+)}(\bar{\Lambda}_i)}(v_{(+,\cdots,+)} \otimes v_{(+,\cdots,+)}) = q^{-\frac{n}{4}}(v_{(+,\cdots,+)} \otimes v_{(+,\cdots,+)}).$$

On the other hand we obtain

$$\begin{aligned}\beta_n^{(+,+)}(z)R_n^{(+,+)}(z)v_{(+,\cdots,+)}\otimes v_{(+,\cdots,+)} &= \beta_n^{(+,+)}(z)v_{(+,\cdots,+)}\otimes v_{(+,\cdots,+)} \\ &= cv_{(+,\cdots,+)}\otimes v_{(+,\cdots,+)} + (\text{higher degree of } z),\end{aligned}$$

then we see

$$c = q^{-\frac{n}{4}}.$$

8. Appendix B

We describe calculation of two point function of the case 1 in Section 4.3. (The other cases can be calculated similarly.) In this case, n is an even integer and two point function is given by

$$\langle \Phi_{\Lambda_n}^{\Lambda_0\ (V^{(n)})_2}(z_2)\Phi_{\Lambda_0}^{\Lambda_n\ (V^{(n)})_1}(z_1)\rangle.$$

We denote this two point function by $\Psi_n(z_1,z_2)$. Here we remark that (cf. [1])

$$\Psi_n(z_1,z_2)\in V^{(+)}\otimes V^{(+)}\otimes (z_1/z_2)^{\Delta_{\Lambda_n}-\Delta_{\Lambda_0}}\otimes \mathbb{Q}(q)[[z_1/z_2]]. \tag{75}$$

We can determine the explicit form of $\Psi_n(z_1,z_2)$ by using Proposition 4.3 and the following lemma.

Lemma 8.1. *If a vector v_n in $V^{(+)}\otimes V^{(+)}$ satisfies*

$$(\pi_{z_1}^{(+)}\otimes\pi_{z_2}^{(+)})\triangle'(e_i)^{\langle h_i,\nu\rangle+1}v_n = 0, \quad \textit{and} \quad \mathrm{wt}\, v_n = 0,$$

for all $i=0,1,\cdots,n$, then v_n is uniquely determined up to multiple constant. More exactly v_n is given by

$$v_n = v_{+-}\otimes v_{n-1} + (-q)^{-n+1}v_{-+}\otimes\sigma_{n-1}v_{n-1}, \quad v_1 = v_{+-}.$$

Combining this lemma with Proposition 4.3, we can find

$$\Psi_n(z_1,z_2) = \psi_n(z_1,z_2)v_n, \tag{76}$$

for some scalar function $\psi_n(z_1,z_2)$. As being described in [2], the two point function can be written as

$$\Psi_n(z_1,z_2) = z_1^{\Delta_{\Lambda_n}-\Delta_{\Lambda_0}}z_2^{\Delta_{\Lambda_0}-\Delta_{\Lambda_n}}\tilde{\Psi}_n(z_1/z_2).$$

Then the scalar function $\psi(z_1,z_2)$ can be also written as

$$\psi_n(z_1,z_2) = z_1^{\Delta_{\Lambda_n}-\Delta_{\Lambda_0}}z_2^{\Delta_{\Lambda_0}-\Delta_{\Lambda_n}}\tilde{\psi}_n(z_1/z_2),$$

for some scalar function $\tilde{\psi}_n(z)$. Substituting (76) to the q-KZ equation we have

$$(pz)^{\Delta_{\Lambda_n}}\tilde{\psi}_n(pz)v_n = z^{\Delta_{\Lambda_n}}\tilde{\psi}_n(z)\beta_n^{(+,+)}(pz)\bar{R}_n^{(+,+)}(pz)(q^{-2\bar{\rho}}\otimes 1)v_n, \tag{77}$$

where $z=z_1/z_2$.

Lemma 8.2.

$$\bar{R}_n^{(+,+)}(z)(q^{-2\bar{\rho}}\otimes 1)v_n = f_n(z)v_n,$$

$$f_n(z) = q^{\frac{1}{2}n^2}\prod_{i=1}^{\frac{n}{2}}\frac{(1-q^{-4i+2}z)}{(1-q^{4i-2}z)}.$$

Proof. Using the recursive relation

$$v_n = v_{+-}\otimes v_{n-1} + (-q)^{-n+1}v_{-+}\otimes \sigma_{n-1}v_{n-1},$$

we can easily show the following formula:

$$X_{n-1}^{(+,-)}(q^{-2\bar{\rho}'}\otimes 1)v_{n-1} = [n-1]_q v_{n-1}, \tag{78}$$

where $X_{n-1}^{(+,-)}$ is given by (20) and $\bar{\rho}' = (2n-4)\omega_2 + (2n-6)\omega_3 + \cdots + 2\omega_{n-1}$ (see Section 2.1). By using the recursive relation of the R-matrices in (18), we find

$$\begin{aligned}\bar{R}_n^{(+,+)}(z)(q^{-2\bar{\rho}}\otimes 1)v_n =& q^{-n+1}v_{+-}\otimes a(z)\bar{R}_{n-1}^{(+,-)}(q^2z)(q^{-2\bar{\rho}'}\otimes 1)v_{n-1}\\ &+ q^{-n+1}v_{-+}\otimes zb(z)\bar{R}_{n-1}^{(-,+)}(q^2z)\sigma_{n-1}X_{n-1}^{(+,-)}(q^{-2\bar{\rho}'}\otimes 1)v_{n-1}\\ &+ (-1)^{-n+1}v_{+-}\otimes b(z)\bar{R}_{n-1}^{(+,-)}(q^2z)\sigma_{n-1}X_{n-1}^{(-,+)}(q^{-2\bar{\rho}'}\otimes 1)\sigma_{n-1}v_{n-1}\\ &+ (-1)^{-n+1}v_{-+}\otimes a(z)\bar{R}_{n-1}^{(-,+)}(q^2z)(q^{-2\bar{\rho}'}\otimes 1)\sigma_{n-1}v_{n-1}.\end{aligned}$$

Combining the above formula (78) with

$$\sigma_{n-1}\bar{R}_{n-1}^{(+,-)}(z) = \bar{R}_{n-1}^{(-,+)}(z)\sigma_{n-1}, \quad \sigma_{n-1}X_{n-1}^{(+,-)} = X_{n-1}^{(-,+)}\sigma_{n-1},$$

and

$$\sigma_{n-1}(q^{-2\bar{\rho}'}\otimes 1) = (q^{-2\bar{\rho}'}\otimes 1)\sigma_{n-1},$$

we have

$$\begin{aligned}\bar{R}_n^{(+,+)}(z)(q^{-2\bar{\rho}}\otimes 1)v_n =& q^n\frac{1-q^{-2n+2}z}{1-q^2z}\{v_{+-}\otimes \bar{R}_{n-1}^{(+,-)}(q^2z)(q^{-2\bar{\rho}'}\otimes 1)v_{n-1}\\ &+ (-q)^{-n+1}v_{-+}\otimes\sigma_{n-1}\bar{R}_{n-1}^{(+,-)}(q^2z)(q^{-2\bar{\rho}'}\otimes 1)v_{n-1}\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\bar{R}_n^{(+,-)}(z)(q^{-2\bar{\rho}}\otimes 1)v_n \;=\;& q^{n-1}\frac{1-q^{-2n+2}z}{1-z}\{v_{+-}\otimes \bar{R}_{n-1}^{(+,+)}(q^2z)(q^{-2\bar{\rho}'}\otimes 1)v_{n-1}\\ &+(-q)^{-n+1}v_{-+}\otimes\sigma_{n-1}\bar{R}_{n-1}^{(+,+)}(q^2z)(q^{-2\bar{\rho}'}\otimes 1)v_{n-1}\},\end{aligned}$$

for any odd integer n. We obtain the eigenvalue $f_n(z)$ by using these recursive relation inductively. Q.E.D

By using this lemma, the q-KZ equation (77) can be reduced to the following difference equation:

$$\tilde{\psi}_n(pz) = p^{-\Delta_{\Lambda_n}} \beta_n^{(+,+)}(pz) f_n(pz) \tilde{\psi}_n(z).$$

Then we have

$$\begin{aligned}
\tilde{\psi}_n(z) &= c \prod_{j=1}^{\infty} \{ p^{\Delta_{\Lambda_n}} f_n(p^j z)^{-1} \beta_n(p^j z)^{-1} \} \\
&= c \prod_{j=1}^{\infty} \{ \prod_{i=1}^{\frac{n}{2}} \frac{(1-q^{4i-2}p^j z)}{(1-q^{-4i+2}p^j z)} \prod_{i=1}^{\frac{n}{2}} \frac{(q^{4n-4i}p^j z;\xi^2)_\infty (q^{4i-4}p^j z;\xi^2)_\infty}{(q^{4n-4i-2}p^j z;\xi^2)_\infty (q^{4i-2}p^j z;\xi^2)_\infty} \} \\
&= c \prod_{j=1}^{\infty} \{ \prod_{i=1}^{\frac{n}{2}} \frac{(q^{-4i+2}p^{j+1} z;\xi^2)_\infty (q^{4i-4}p^j z;\xi^2)_\infty}{(q^{-4i+2}p^j z;\xi^2)_\infty (q^{4i-4}p^{j+1} z;\xi^2)_\infty} \} \\
&= c \prod_{i=1}^{\frac{n}{2}} \frac{(q^{4i-4}pz;\xi^2)_\infty}{(q^{-4i+2}pz;\xi^2)_\infty}.
\end{aligned}$$

By the property in (75), we see that

$$\tilde{\psi}_n(z) \in \mathbb{Q}(q)[[z]].$$

Then c is a constant. This constant c can be determined by the constant term of the two point function $\tilde{\Psi}_n(z_1/z_2) = \tilde{\psi}_n(z_1/z_2)v_n$. It is easy to know this constant term from the normalization of vertex operators (see Section 4.1) and we find

$$c = (-q)^{\frac{1}{2}n(n-1)}.$$

Then we obtain the explicit form of the two point function.

9. Appendix C

We give explicit forms of isomorphisms of $U_q(\mathfrak{g})$-module in (25)

$$C_{\pm}^{(k)} \ : \ V_{z\xi^{\mp}}^{(k)} \longrightarrow V_z^{(k)*a^{\pm 1}} \quad (1 \le k \le n-2),$$

though the existence of these isomorphisms is proved in [5].

Let us recall that there exist the following isomorphisms of $U_q(\mathring{\mathfrak{g}})$-modules

$$V_z^{(k)} \simeq V_{\bar{\Lambda}_k} \oplus V_{\bar{\Lambda}_{k-2}} \oplus \cdots \oplus (V_{\bar{\Lambda}_1} \text{ or } V_0) \ \ (k = 1, 2, \ldots n-2), \tag{79}$$

where $V_{\bar{\Lambda}_i}$ is denoted the irreducible $U_q(\mathring{\mathfrak{g}})$-module with highest weight $\bar{\Lambda}_i$ (cf. [6]). Here we know that if two irreducible finite-dimensional representations of $U_q(\mathring{\mathfrak{g}})$ have the same highest weight then they are

isomorphic. For $i = 1, 2, \dots n-2$, $V_{\bar{\Lambda}_i}$ and $(V_{\bar{\Lambda}_i})^{*a^{\pm 1}}$ have the same highest weight, then there exists an isomorphism

$$\bar{C}_{i,\pm} : V_{\bar{\Lambda}_i} \longrightarrow (V_{\bar{\Lambda}_i})^{*a^{\pm 1}},$$

and this isomorphism is unique up to multiple constant. Thanks to the isomorphism (79), we can write $C_{\pm}^{(k)}$ as a linear combination of $\bar{C}_{i,\pm}$. Then we normalize $\bar{C}_{i,\pm}$ and express the isomorphism $C_{\pm}^{(k)}$ by using them. Let us recursively define a vector x_m by

$$x_m = v_{+-} \otimes x_{m-1} + (-q)^{m-1} v_{-+} \otimes \sigma_{m-1} x_{m-1}, \quad x_1 = v_{+-},$$

where $x_m \in V_m^{(+)} \otimes V_m^{(\varepsilon)}$ and if m is even (resp. odd) then $\varepsilon = +$, (resp.$-$). By using x_m, we recursively define $y_{n,\pm}^{(m)}$ $(m \leq n)$ by

$$\begin{aligned} y_{n,+}^{(m)} &= v_{++} \otimes y_{n-1,+}^{(m)} && (n > m), \\ y_{n,-}^{(m)} &= v_{--} \otimes \sigma_{n-1} y_{n-1,-}^{(m)} && (n > m), \\ y_{m,+}^{(m)} &= y_{m,-}^{(m)} = x_m && (m = n). \end{aligned}$$

We remark that

$$V^{(k)} = V^{(n)} \otimes V^{(n')} / \ker T^{(k)} \quad (n' = \varphi(n-k)),$$

and the equivalent classes represented by $y_{n,+}^{(m)}$ (resp. $y_{n,-}^{(m)}$) are the highest weight vector (resp. the lowest weight vector) in $U_q(\mathring{\mathfrak{g}})$-module $V_{\bar{\Lambda}_{n-m}}$. Let us normalize $\bar{C}_{i,\pm}$ $(i = 1, 2, \dots, n-2)$ by

$$\bar{C}_{n-m,\pm}(y_{m,+}) = y_{m,-}^{*} \ (m = 2, 3, \dots, n).$$

By using these isomorphisms $\bar{C}_{i,\pm}$ $(i = 1, 2, \dots, n-2)$ we have

$$C_{\pm}^{(k)} = \sum_{m=0}^{[\frac{k}{2}]} k_{m,\pm} \bar{C}_{k-2m,\pm}.$$

By direct calculation we can show

$$k_{m,\pm} \;=\; (-q)^{\pm m(2n-2k-2m+1)} \prod_{i=0}^{m-1} \frac{[2n-2k+2i]_q}{[2i+2]_q} (1 + q^{2n-2k+4i})(1 + q^{2n-2k+4i+2}).$$

References

[1] B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki: Diagonalization of the XXZ Hamiltonian by vertex operators, Comm. Math. Phys., **151**, (1993), 86–153.

[2] M. Idzumi, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima and T. Tokihiro: Quantum affine symmetry in vertex models, Int. J. Mod. Phys., A, **8**, (1993), 1479–1511.

[3] E. Date and M. Okado: Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type $A_n^{(1)}$, Int. J. Mod. Phys., A, **9**, (1994), 399–417.

[4] E. Date M. Jimbo and M. Okado: Crystal base and q-vertex operators, Comm. Math. Phys., **155**, (1993), 47–69.

[5] B. Davies and M. Okado: Excitation spectra of the spin models constructed from quantized affine algebras of type $B_n^{(1)}, D_n^{(1)}$, Int. J. Mod. Phys., A, **11**, (1996), 1975–2017.

[6] M. Okado: Quantum R matrix related to the spin representation of $B_n^{(1)}, D_n^{(1)}$, Comm. Math. Phys., **134**, (1990), 467–486.

[7] S-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki: Affine crystals and vertex models, Int. J. Mod. Phys. A, **7**, Suppl. 1A (1992) 449–484.

[8] O. Babelon, H. J. de Vega and C. M. Viallet: Exact excitation spectra of the $Z_{n+1} \times Z_{n+1}$ generalized Heisenberg model, Nucl. Phys. B, **220**, [FS8] (1983), 283–301.

[9] T. Nakanishi: Fusion, mass and representation theory of the Yangian algebra, Nucl. Phys. B, **439**, (1995), 441–459.

[10] E. Ogievesky, N. Yu. Reshetikhin and P. Wiegmann: The principal chiral field in two dimensions on classical Lie algebras, Nucl. Phys. B, **15**, (1987), 45–98.

[11] V. G. Kac: Infinite dimensional Lie algebras, 3rd ed., Cambridge University Press, Cambridge 1990.